Regular Expressions

Recap from Last Time

Regular Languages

- A language L is called a ***regular language*** if there is a DFA or an NFA for L.
- ***Theorem:*** The following are equivalent:
  - L is a regular language.
  - There is a DFA D where $\mathscr{L}(D) = L$.
  - There is an NFA N where $\mathscr{L}(N) = L$.
- In other words, knowing any one of the above three facts means you know the other two.

Language Concatenation

- If $w \in \Sigma^*$ and $x \in \Sigma^*$, then wx is the ***concatenation*** of w and x.
- If L_1 and L_2 are languages over Σ, the ***concatenation*** of L_1 and L_2 is the language L_1L_2 defined as

$$L_1L_2 = \{ x \mid \exists w_1 \in L_1.\ \exists w_2 \in L_2.\ x = w_1w_2 \}$$

- Example: if $L_1 = \{$ **a**, **ba**, **bb** $\}$ and $L_2 = \{$ **aa**, **bb** $\}$, then

$L_1L_2 = \{$ **aaa**, **abb**, **baaa**, **babb**, **bbaa**, **bbbb** $\}$

Lots and Lots of Concatenation

- Consider the language L = { aa, b }
- LL is the set of strings formed by concatenating pairs of strings in L.

{ aaaa, aab, baa, bb }

- LLL is the set of strings formed by concatenating triples of strings in L.

{ aaaaaa, aaaab, aabaa, aabb, baaaa, baab, bbaa, bbb}

- $LLLL$ is the set of strings formed by concatenating quadruples of strings in L.

{ aaaaaaaa, aaaaaab, aaaabaa, aaaabb, aabaaaa, aabaab, aabbaa, aabbb, baaaaaa, baaaab, baabaa, baabb, bbaaaa, bbaab, bbbaa, bbbb}

Language Exponentiation

- We can define what it means to "exponentiate" a language as follows:

$$L^0 = \{\varepsilon\} \qquad L^{n+1} = LL^n$$

- So, for example, { **aa**, **b** }^{3} is the language

{ **aaaaaa**, **aaaab**, **aabaa**, **aabb**,
baaaa, **baab**, **bbaa**, **bbb**}

The Kleene Closure

- An important operation on languages is the ***Kleene Closure***, which is defined as

$$\boldsymbol{L^* = \{ w \in \Sigma^* \mid \exists n \in \mathbb{N}.\ w \in L^n \}}$$

- Mathematically:

$$\boldsymbol{w \in L^* \quad \textbf{iff} \quad \exists n \in \mathbb{N}.\ w \in L^n}$$

- Intuitively, all possible ways of concatenating zero or more strings in L together, possibly with repetition.

The Kleene Closure

If L = { **a**, **bb** }, then L^* = {

ε,

a, **bb**,

aa, **abb**, **bba**, **bbbb**,

aaa, **aabb**, **abba**, **abbbb**, **bbaa**, **bbabb**, **bbbba**, **bbbbbb**,

…

}

Think of L* as the set of strings you can make if you have a collection of stamps – one for each string in L – and you form every possible string that can be made from those stamps.

Closure Properties

- ***Theorem:*** If L_1 and L_2 are regular languages over an alphabet Σ, then so are the following languages:
  - $L_1 \cup L_2$
  - L_1L_2
  - L_1*
- These (and other) properties are called ***closure properties of the regular languages***.

New Stuff!

Another View of Regular Languages

Rethinking Regular Languages

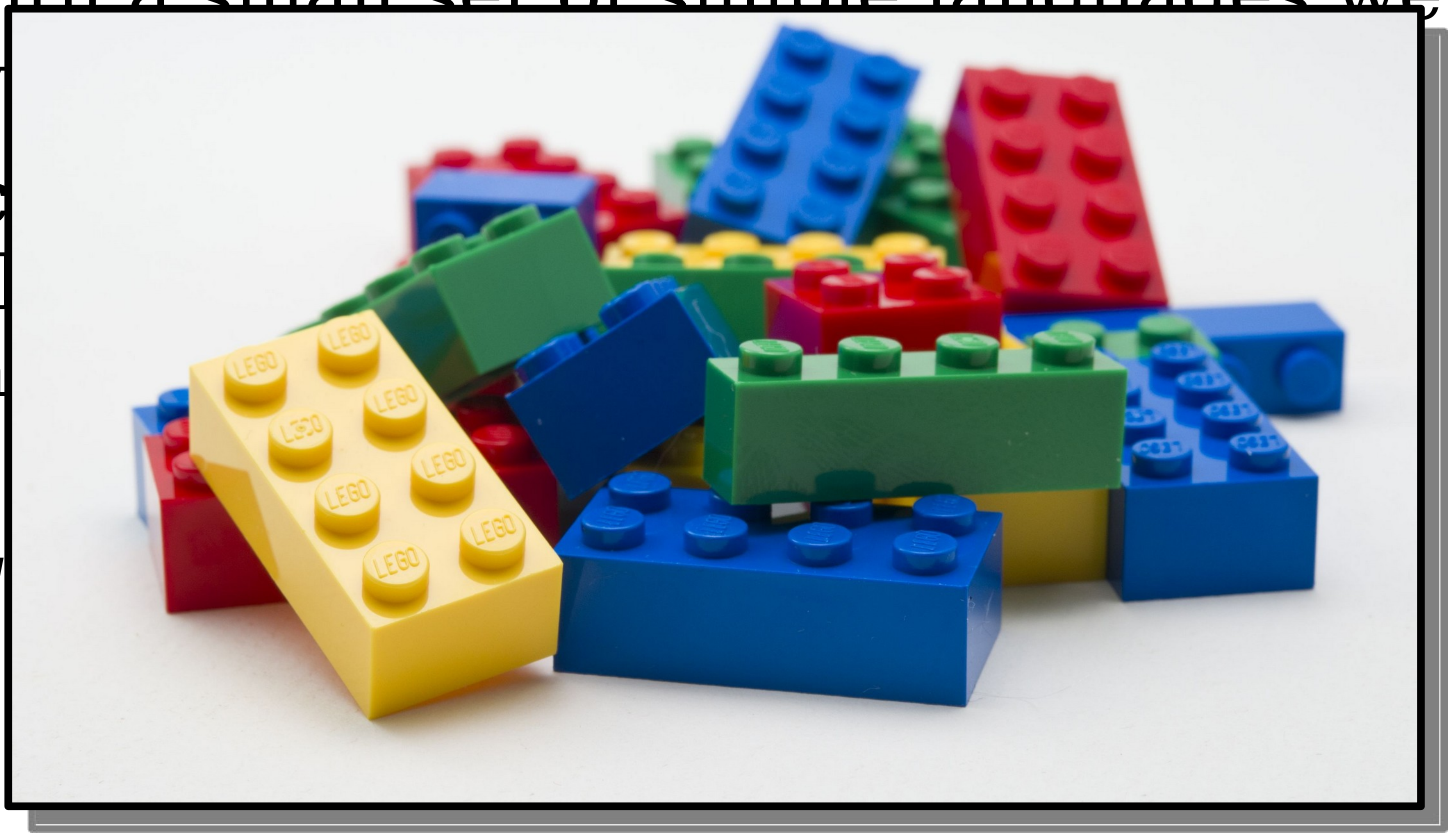
- We currently have several tools for showing a language L is regular:
  - Construct a DFA for L.
  - Construct an NFA for L.
  - Combine several simpler regular languages together via closure properties to form L.
- We have not spoken much of this last idea.

Constructing Regular Languages

- ***Idea:*** Build up all regular languages as follows:
  - Start with a small set of simple languages we already know to be regular.
  - Using closure properties, combine these simple languages together to form more elaborate languages.
- *This is a bottom-up approach to the regular languages.*

Constructing Regular Languages

- ***Idea:*** Build up all regular languages as follows:
  - Start with a small set of simple languages we already
  - Using c simple l elabora
- *This is a regular l*



Regular Expressions

- ***Regular expressions*** are a way of describing a language via a string representation.
- They're used just about everywhere:
  - They're built into the JavaScript language and used for data validation.
  - They're used in the UNIX `grep` and `flex` tools to search files and build compilers.
  - They're employed to clean and scrape data for large-scale analysis projects.
- Conceptually, regular expressions are strings describing how to assemble a larger language out of smaller pieces.

Atomic Regular Expressions

- The regular expressions begin with three simple building blocks.
- The symbol **Ø** is a regular expression that represents the empty language Ø.
- For any **a** ∈ Σ, the symbol **a** is a regular expression for the language {**a**}.
- The symbol **ε** is a regular expression that represents the language {ε}.
  - ***Remember: {ε} ≠ Ø!***
  - ***Remember: {ε} ≠ ε!***

Compound Regular Expressions

- If R_1 and R_2 are regular expressions, $\boldsymbol{R_1R_2}$ is a regular expression for the *concatenation* of the languages of R_1 and R_2.
- If R_1 and R_2 are regular expressions, $\boldsymbol{R_1 \cup R_2}$ is a regular expression for the *union* of the languages of R_1 and R_2.
- If R is a regular expression, $\boldsymbol{R^*}$ is a regular expression for the *Kleene closure* of the language of R.
- If R is a regular expression, $\boldsymbol{(R)}$ is a regular expression with the same meaning as R.

Operator Precedence

- Here's the operator precedence for regular expressions:

$$(R)$$

$$R^*$$

$$R_1R_2$$

$$R_1 \cup R_2$$

- So **ab*c∪d** is parsed as **((a(b*))c)∪d**

Regular Expression Examples

- The regular expression **trick∪treat** represents the language

  { **trick**, **treat** }.

- The regular expression **booo*** represents the regular language

  { **boo**, **booo**, **boooo**, … }.

- The regular expression **candy!(candy!)*** represents the regular language

  { **candy!**, **candy!candy!**, **candy!candy!candy!**, … }.

Regular Expressions, Formally

- The ***language of a regular expression*** is the language described by that regular expression.
- Formally:
  - $\mathscr{L}(\boldsymbol{\varepsilon}) = \{\varepsilon\}$
  - $\mathscr{L}(\boldsymbol{\emptyset}) = \emptyset$
  - $\mathscr{L}(\mathbf{a}) = \{\mathbf{a}\}$
  - $\mathscr{L}(R_1 R_2) = \mathscr{L}(R_1)\ \mathscr{L}(R_2)$
  - $\mathscr{L}(R_1 \cup R_2) = \mathscr{L}(R_1) \cup \mathscr{L}(R_2)$
  - $\mathscr{L}(R^{*}) = \mathscr{L}(R)^{*}$
  - $\mathscr{L}((R)) = \mathscr{L}(R)$

Worthwhile activity: Apply this recursive definition to

a(b∪c)((d))

and see what you get.

Designing Regular Expressions

- Let $\Sigma = \{$**`a`**$, $**`b`**$\}$.
- Let $L = \{ w \in \Sigma^* \mid w$ contains **`aa`** as a substring $\}$.

Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{ w \in \Sigma^* \mid w$ contains **aa** as a substring $\}$.

(a ∪ b)*aa(a ∪ b)*

Designing Regular Expressions

- Let $\Sigma = \{$`a`, `b`$\}$.
- Let $L = \{ w \in \Sigma^* \mid w$ contains `aa` as a substring $\}$.

`(a ∪ b)*aa(a ∪ b)*`

Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{ w \in \Sigma^* \mid w$ contains **aa** as a substring $\}$.

(a ∪ b)*aa(a ∪ b)*

bbabbbaabab

aaaa

bbbbabbbbaabbbb

Designing Regular Expressions

- Let Σ = {**a**, **b**}.
- Let L = { $w \in \Sigma^*$ | w contains **aa** as a substring }.

(a ∪ b)*aa(a ∪ b)*

bbabbbaabab

aaaa

bbbbbabbbbaabbbbb

Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{ w \in \Sigma^* \mid w$ contains **aa** as a substring $\}$.

Σ*aaΣ*

bbabbbaabab

aaaa

bbbbbabbbbaabbbbb

Designing Regular Expressions

- Let $\Sigma = \{\texttt{a}, \texttt{b}\}$.
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$.

Designing Regular Expressions

Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$.

Let $L = \{\ w \in \Sigma^* \mid |w| = 4\ \}$.

The length of a string w is denoted $|w|$

Designing Regular Expressions

- Let $\Sigma = \{\texttt{a}, \texttt{b}\}$.
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$.

Write a regex for this language.

Answer at
https://pollev.com/cs103aut23

Designing Regular Expressions

- Let $\Sigma = \{\texttt{a}, \texttt{b}\}$.
- Let $L = \{\ w \in \Sigma^* \mid |w| = 4\ \}$.

$$\Sigma\Sigma\Sigma\Sigma$$

Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$.

$$\Sigma\Sigma\Sigma\Sigma$$

Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$.

$\Sigma\Sigma\Sigma\Sigma$

aaaa
baba
bbbb
baaa

Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$.

$\Sigma\Sigma\Sigma\Sigma$

aaaa
baba
bbbb
baaa

Designing Regular Expressions

- Let $\Sigma = \{\texttt{a}, \texttt{b}\}$.
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$.

Σ^4

aaaa
baba
bbbb
baaa

Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$.

Σ^4

aaaa
baba
bbbb
baaa

Designing Regular Expressions

- Let Σ = {**a**, **b**}.
- Let L = { $w \in \Sigma^*$ | w contains at most one **a** }.

Here are some candidate regular expressions for the language L. Which of these are correct?

Σ*aΣ*
b*ab* ∪ b*
b*(a ∪ ε)b*
b*a*b* ∪ b*
b*(a* ∪ ε)b*

Answer at ***https://pollev.com/cs103aut23***

Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{ w \in \Sigma^* \mid w$ contains at most one **a** $\}$.

b*(a ∪ ε)b*

Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{ w \in \Sigma^* \mid w$ contains at most one **a** $\}$.

b*(a ∪ ε)b*

Designing Regular Expressions

- Let Σ = {**a**, **b**}.
- Let $L = \{ w \in \Sigma^* \mid w$ contains at most one **a** $\}$.

b*(a ∪ ε)b*

bbbbabbb
bbbbbb
abbb
a

Designing Regular Expressions

- Let Σ = {**a**, **b**}.
- Let L = { $w \in \Sigma^*$ | w contains at most one **a** }.

b*(a ∪ ε)b*

bbbbabbb
bbbbbb
abbb
a

Designing Regular Expressions

- Let Σ = {a, b}.
- Let $L = \{ w \in \Sigma^* \mid w$ contains at most one a $\}$.

b*a?b*

bbbbabbb
bbbbbb
abbb
a

A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com

A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com

A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

aa*

cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com

A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

aa*

cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com

A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

```
aa* (.aa*)*
```

```
cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com
```

A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

```
aa* (.aa*)*
```

```
cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com
```

A More Elaborate Design

- Let $\Sigma = \{$ **a**, **.**, **@** $\}$, where **a** represents "some letter."
- Let's make a regex for email addresses.

```
aa* (.aa*)* @
```

```
cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com
```

A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

aa* (.aa*)* @

cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com

A More Elaborate Design

- Let Σ = { `a`, `.`, `@` }, where `a` represents "some letter."
- Let's make a regex for email addresses.

```
aa* (.aa*)* @ aa*.aa*
```

```
cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com
```

A More Elaborate Design

- Let Σ = { `a`, `.`, `@` }, where `a` represents "some letter."
- Let's make a regex for email addresses.

```
aa* (.aa*)* @ aa*.aa*
```

```
cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com
```

A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

aa* (.aa*)* @ aa*.aa* (.aa*)*

cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com

A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents “some letter.”
- Let's make a regex for email addresses.

aa* (.aa*)* @ aa*.aa* (.aa*)*

cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com

A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

```
a+ (.aa*)* @ aa*.aa* (.aa*)*
```

```
cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com
```

A More Elaborate Design

- Let Σ = { a, ., @ }, where a represents "some letter."
- Let's make a regex for email addresses.

$\mathtt{a^{+}\ (.aa^{*})^{*}\ @\ aa^{*}.aa^{*}\ (.aa^{*})^{*}}$

cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com

A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

$$\mathtt{a}^{+}\ (\mathtt{.a}^{+})^{*}\ \mathtt{@}\ \mathtt{a}^{+}\ \mathtt{.a}^{+}\ (\mathtt{.a}^{+})^{*}$$

cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com

A More Elaborate Design

- Let Σ = { a, ., @ }, where a represents "some letter."
- Let's make a regex for email addresses.

$$a^+ \ (.a^+)^* \ @ \ a^+ .a^+ \ (.a^+)^*$$

cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com

A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

$$a^+ \; (.a^+)^* \; @ \; a^+ \; .a^+ \; (.a^+)^*$$

cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com

A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

$\texttt{a}^+ \ (\texttt{.a}^+)^* \ \texttt{@} \ \texttt{a}^+ \ \texttt{.a}^+ \ (\texttt{.a}^+)^*$

cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com

A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

```
a⁺ (.a⁺)* @ a⁺ (.a⁺)⁺
```

$a^+ \; (.a^+)^* \; @ \; a^+ \; (.a^+)^+$

cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com

A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

$$\mathtt{a^+ \;\; (.a^+)^* \;\; @ \;\; a^+ \qquad (.a^+)^+}$$

cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com

A More Elaborate Design

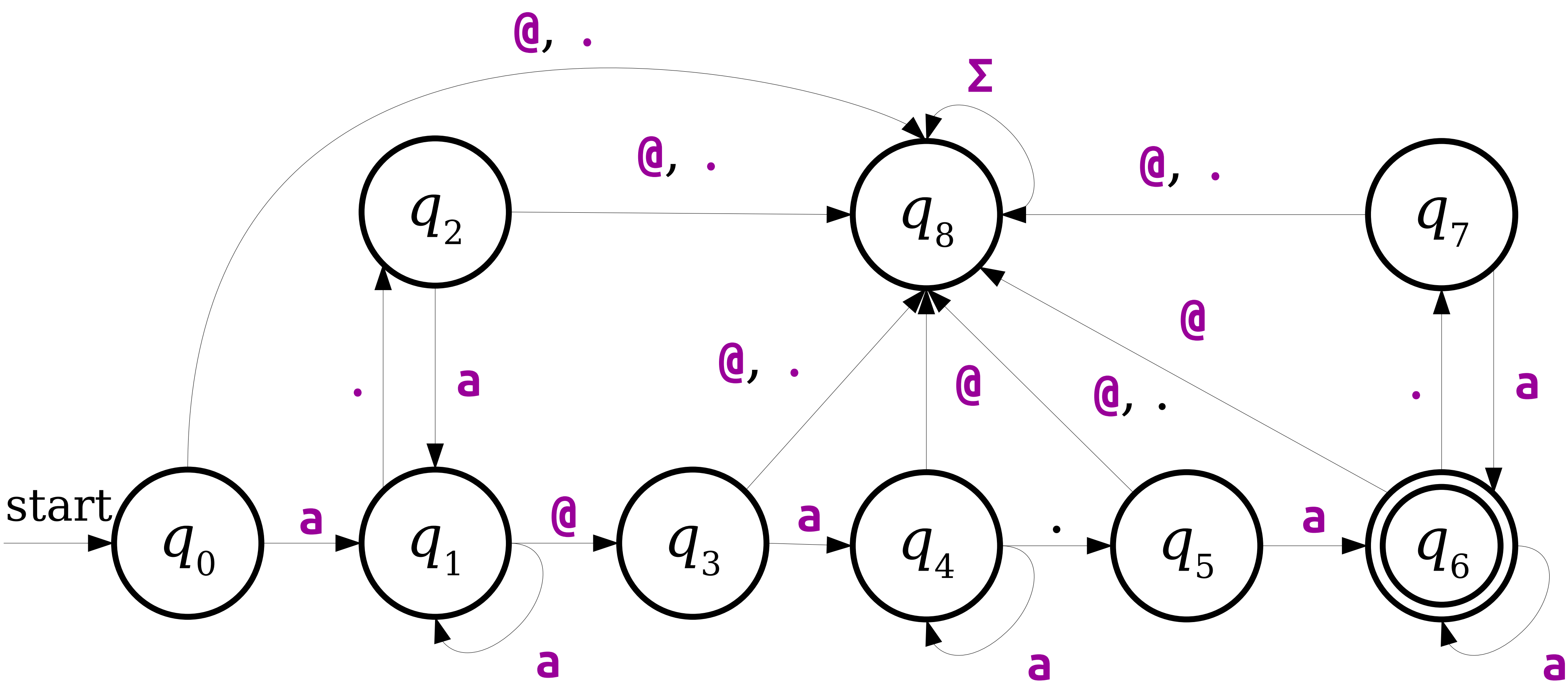
- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

$$a^+ (.a^+)^* @ a^+ (.a^+)^+$$

cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com

For Comparison

a⁺(.a⁺)*@a⁺(.a⁺)⁺



Shorthand Summary

- $\mathbf{R}^n$ is shorthand for **RR** … **R** (*n* times).
  - Edge case: define $\mathbf{R}^0$ = ε.
- **Σ** is shorthand for "any character in Σ."
- **R?** is shorthand for **(R ∪ ε)**, meaning "zero or one copies of *R*."
- $\mathbf{R}^+$ is shorthand for **RR***, meaning "one or more copies of *R*."

Time-Out for Announcements!

Second Midterm Logistics

- Our second midterm exam is next ***Tuesday, November 14th*** from ***7PM – 10PM***. Locations are the same as the first exam and are divvied up by last (family) name:
  - `A – P`: Go to Bishop Auditorium.
  - `Q – Z`: Go to 200-002.
- Topic coverage is primarily lectures 06 – 13 (functions through induction) and PS3 – PS5. Finite automata and onward won't be tested here.
  - Because the material is cumulative, topics from PS1 – PS2 and Lectures 00 – 05 are also fair game.
- Students with OAE accommodations: you should hear from us by the end of the day with room information. If not, contact us tomorrow morning.

Preparing for the Exam

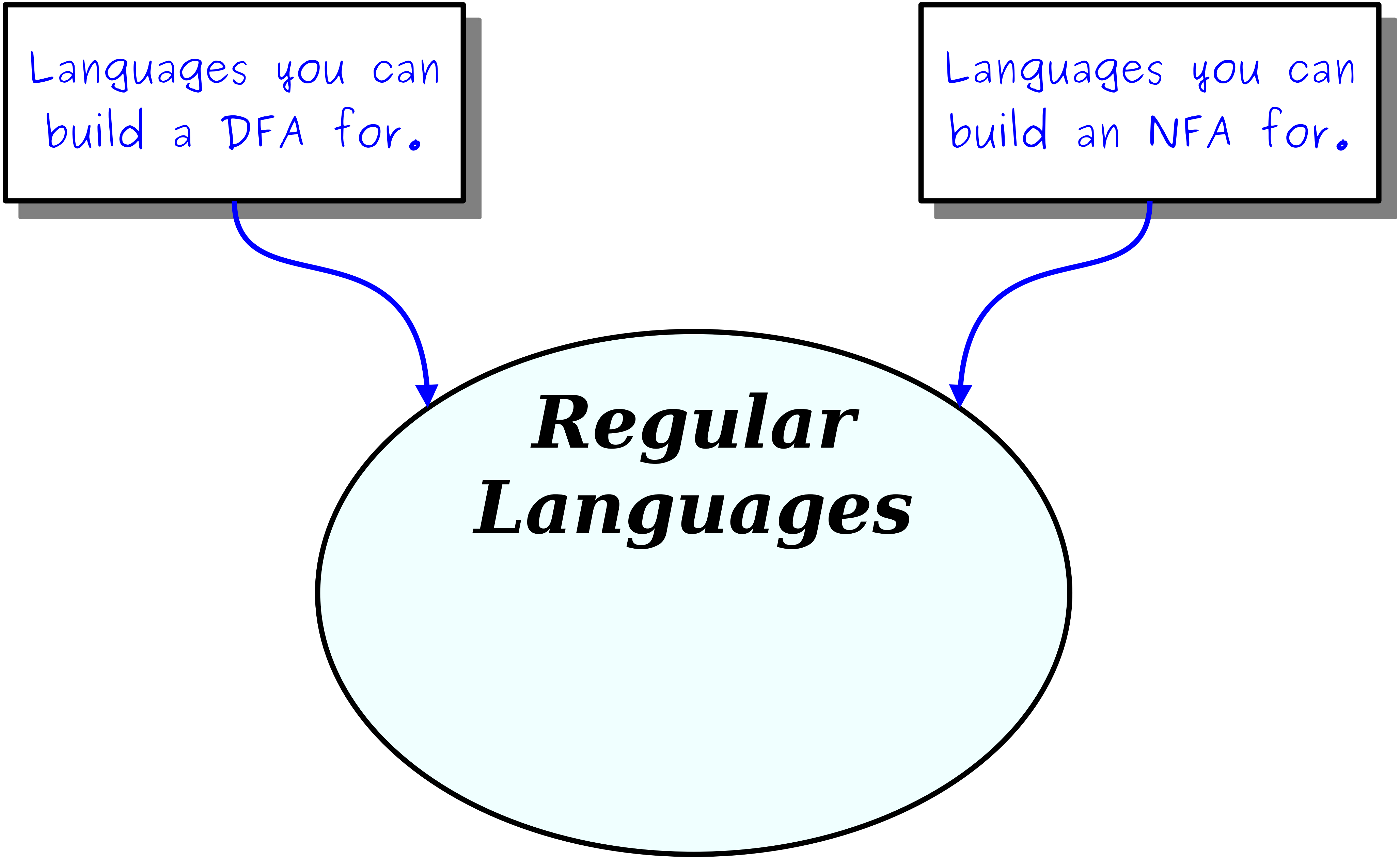
- The top skills that will serve you well on this exam:
  - ***Knowing how to set up a proof***. This is a recurring theme across functions, sets, graphs, pigeonhole, and induction.
  - ***Distinguishing between assuming and proving***. This similarly cuts across all of these topics.
  - ***Reading new definitions***. This is at the heart of mathematical reasoning.
  - ***Writing proofs in line with definitions***. Folks often ask about whether they're being rigorous enough. Often "rigorous enough" simply means "following what the definitions say."
- Keith's personal recommendation: when working through practice problems, pay super extra close attention to these areas.

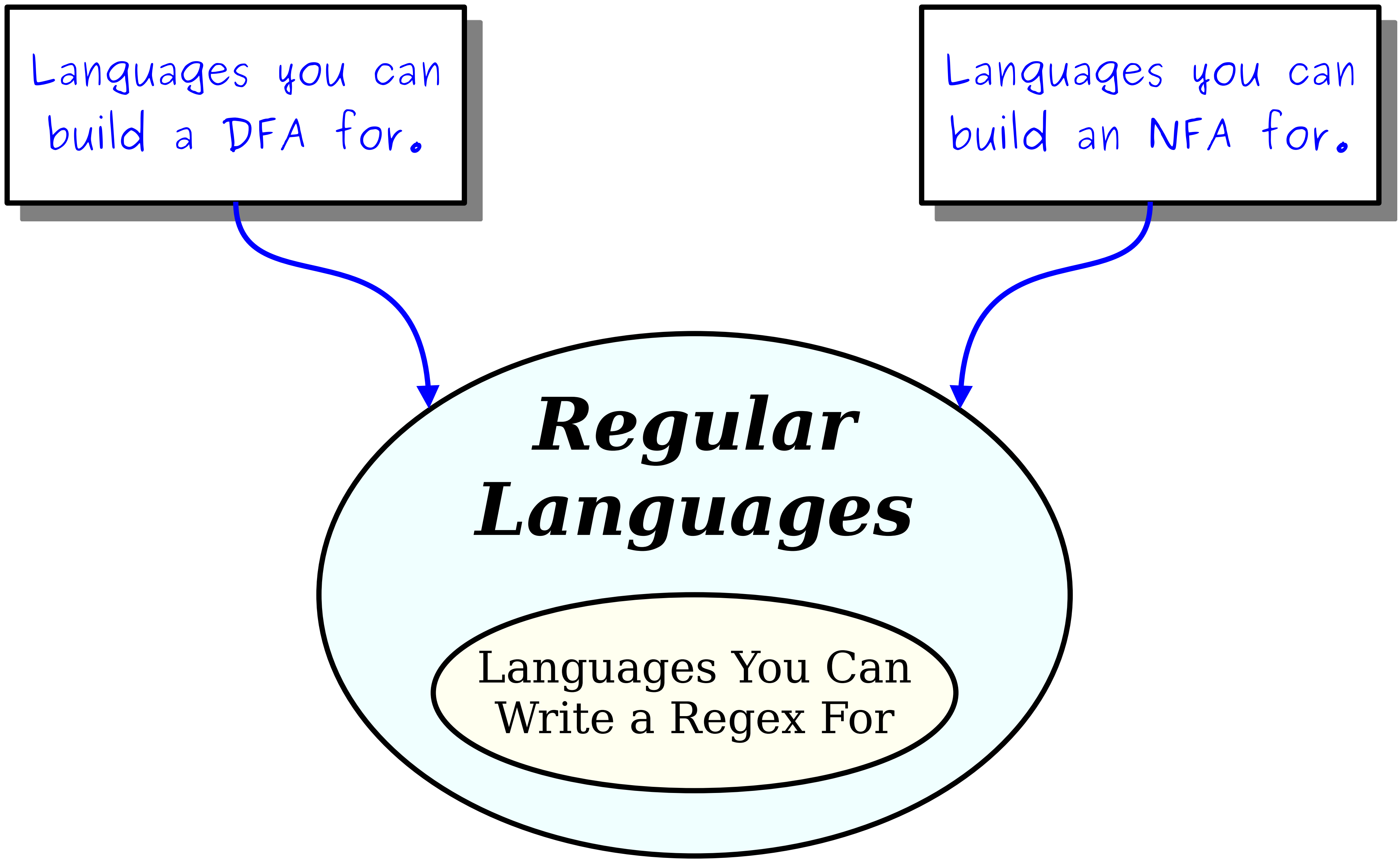
Preparing for the Exam

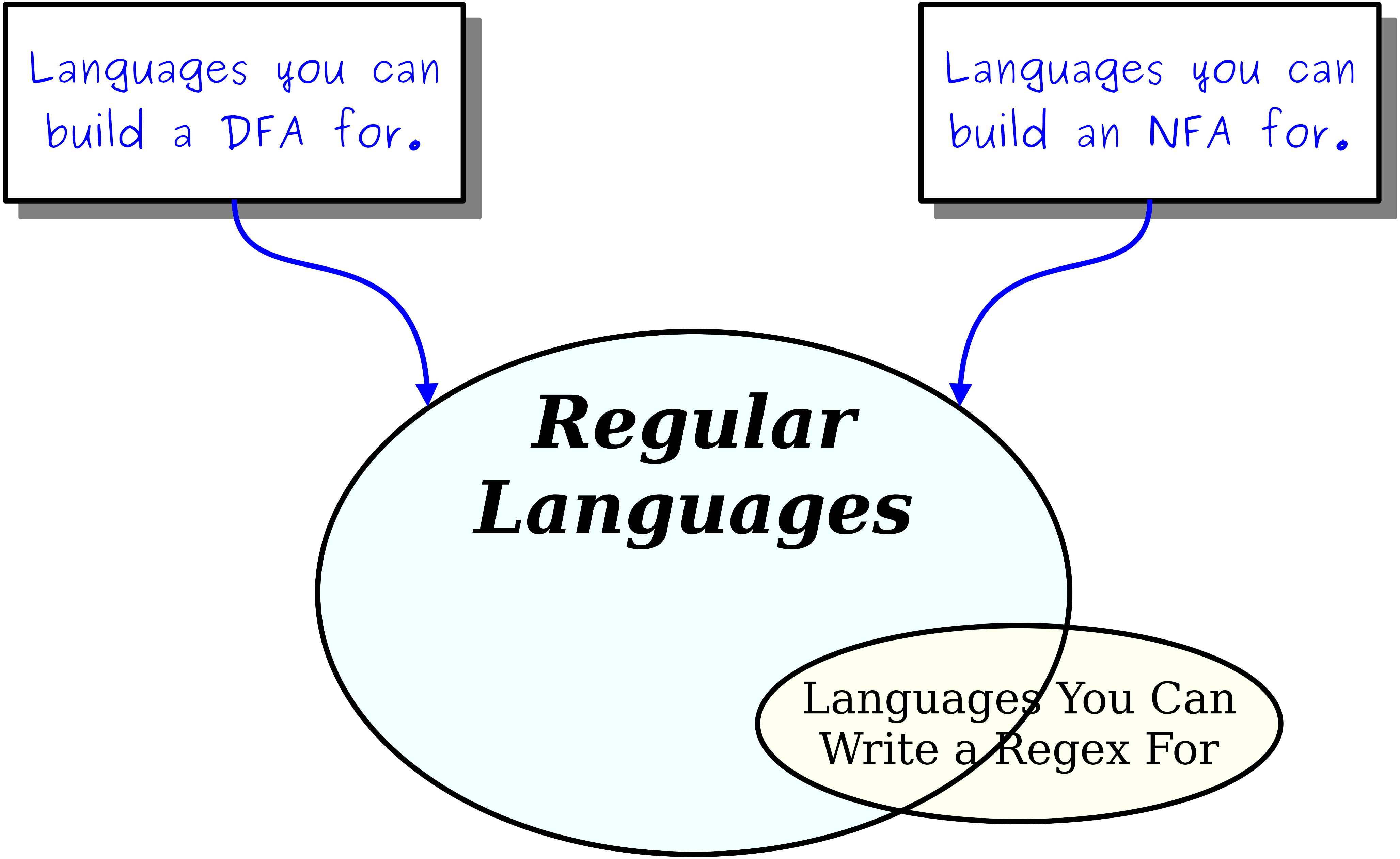
- As with the first midterm exam, we've posted a bunch of practice exams on the course website.
  - There are ten practice exams (yes, really!). We realistically don't expect anyone to complete them all. They're there to give you a feeling of what the exam might look like.
- Some general notes on preparing:
  - Q5 and Q6 on the current problem set, while technically on topics that aren't covered on the midterm, are great practice for the sorts of reasoning you'll need on the exam.
  - ***Keep the TAs in the loop when studying***. Ask for feedback on any proofs you write when getting ready for the exam.
  - Don't skip on biological care and maintenance. Exams can be stressful, but please make time for basic things like showering, eating, etc. and for self-care in whatever form that takes for you.
- ***You can do this***. Best of luck on the exam!

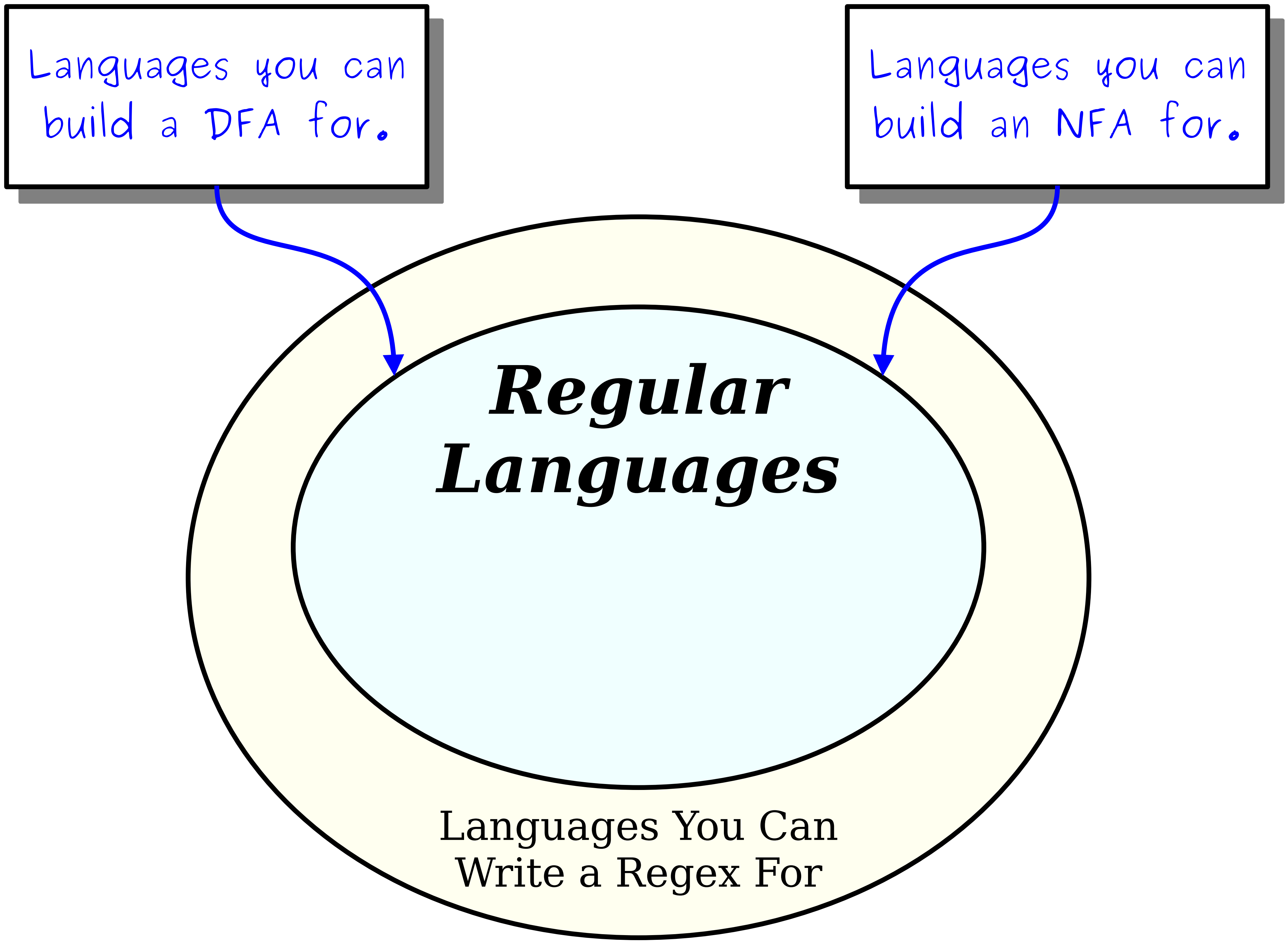
Back to CS103!

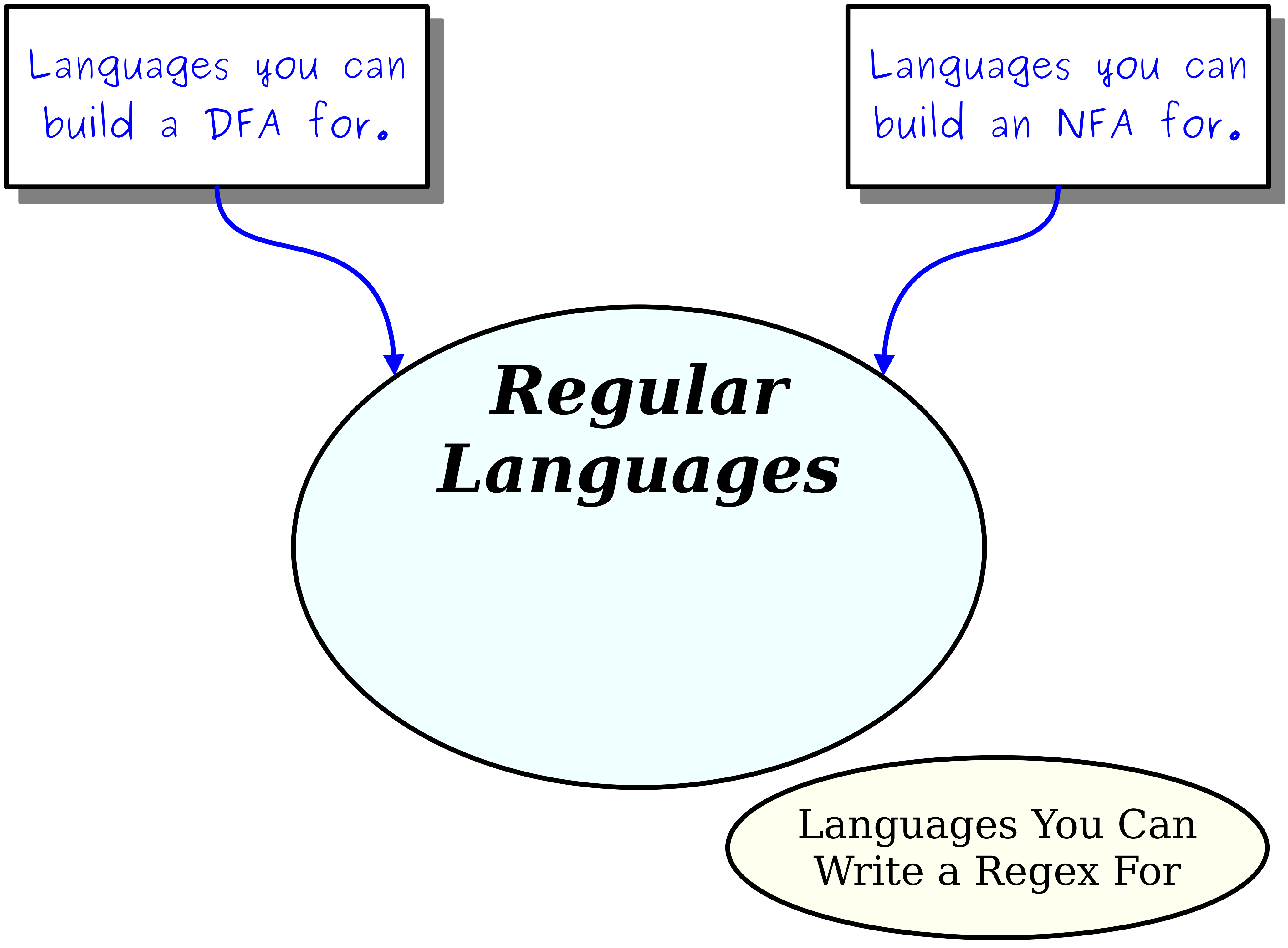
The Lay of the Land

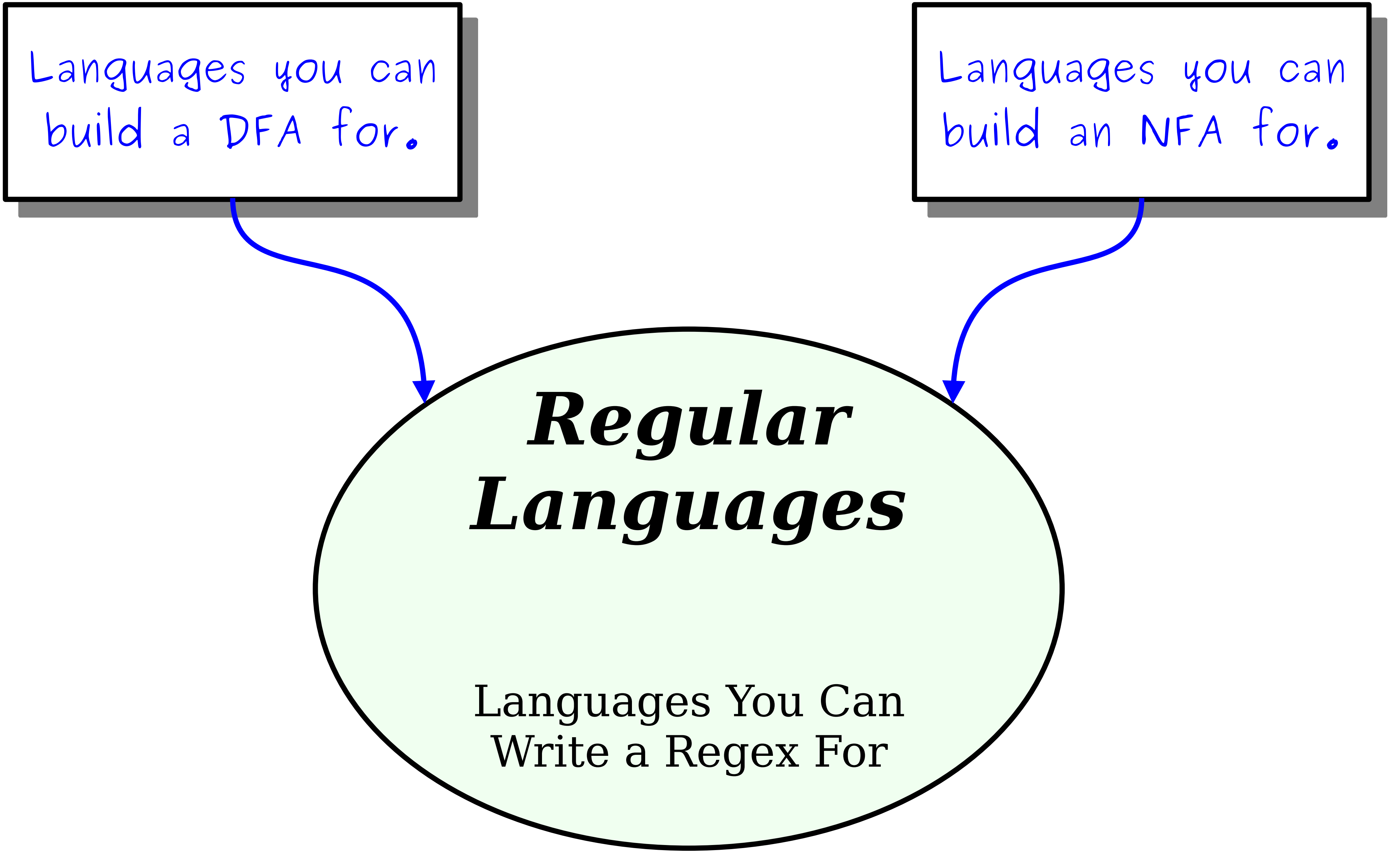

Languages you can build a DFA for.
Languages you can build an NFA for.
Regular Languages


Languages you can build a DFA for.
Languages you can build an NFA for.
Regular Languages
Languages You Can Write a Regex For


Languages you can build a DFA for.
Languages you can build an NFA for.
Regular Languages
Languages You Can Write a Regex For


Languages you can build a DFA for.
Languages you can build an NFA for.
Regular Languages
Languages You Can Write a Regex For


Languages you can build a DFA for.
Languages you can build an NFA for.
Regular Languages
Languages You Can Write a Regex For


Languages you can build a DFA for.
Languages you can build an NFA for.
Regular Languages
Languages You Can Write a Regex For

The Power of Regular Expressions

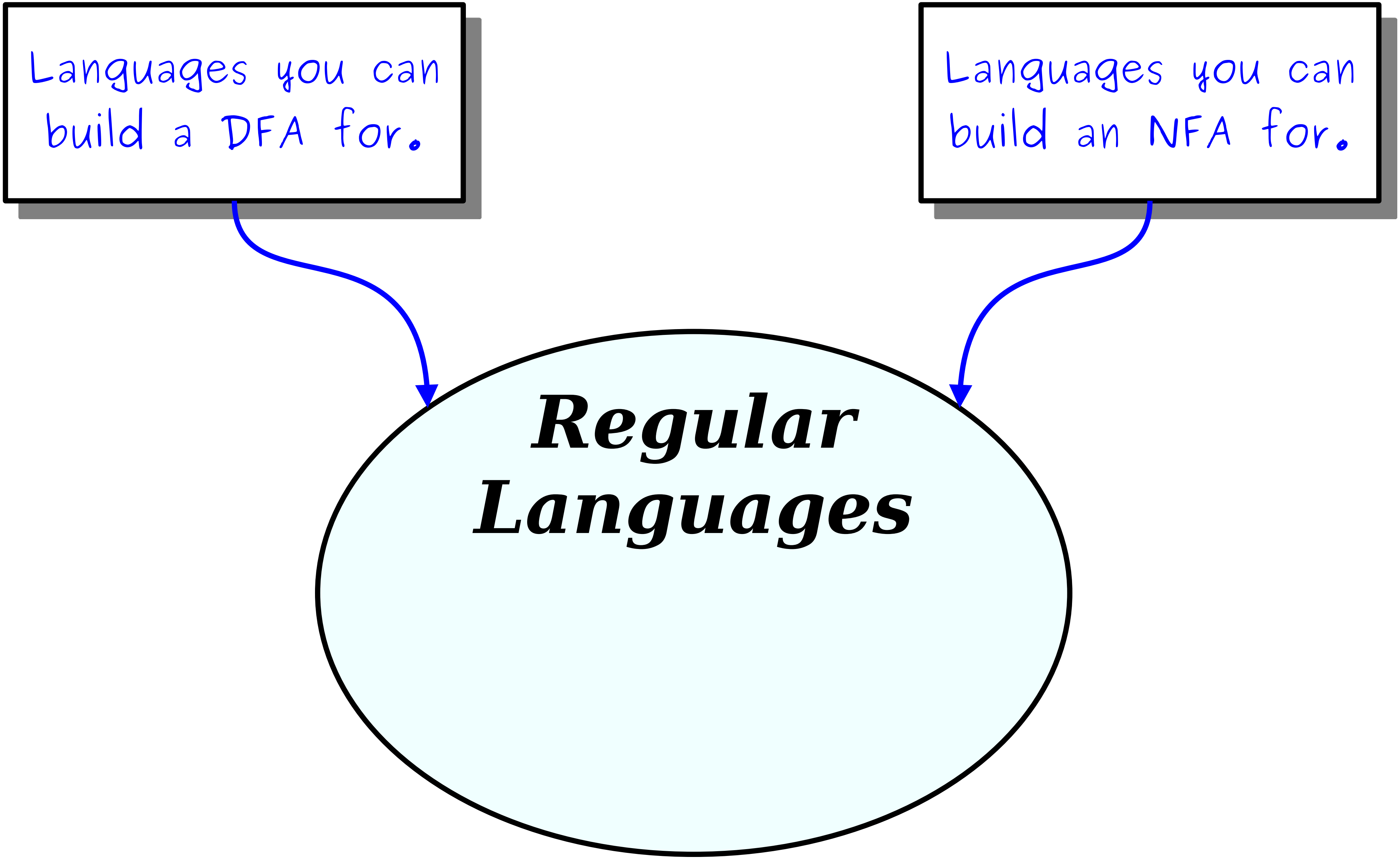
Theorem: If R is a regular expression, then $\mathscr{L}(R)$ is regular.

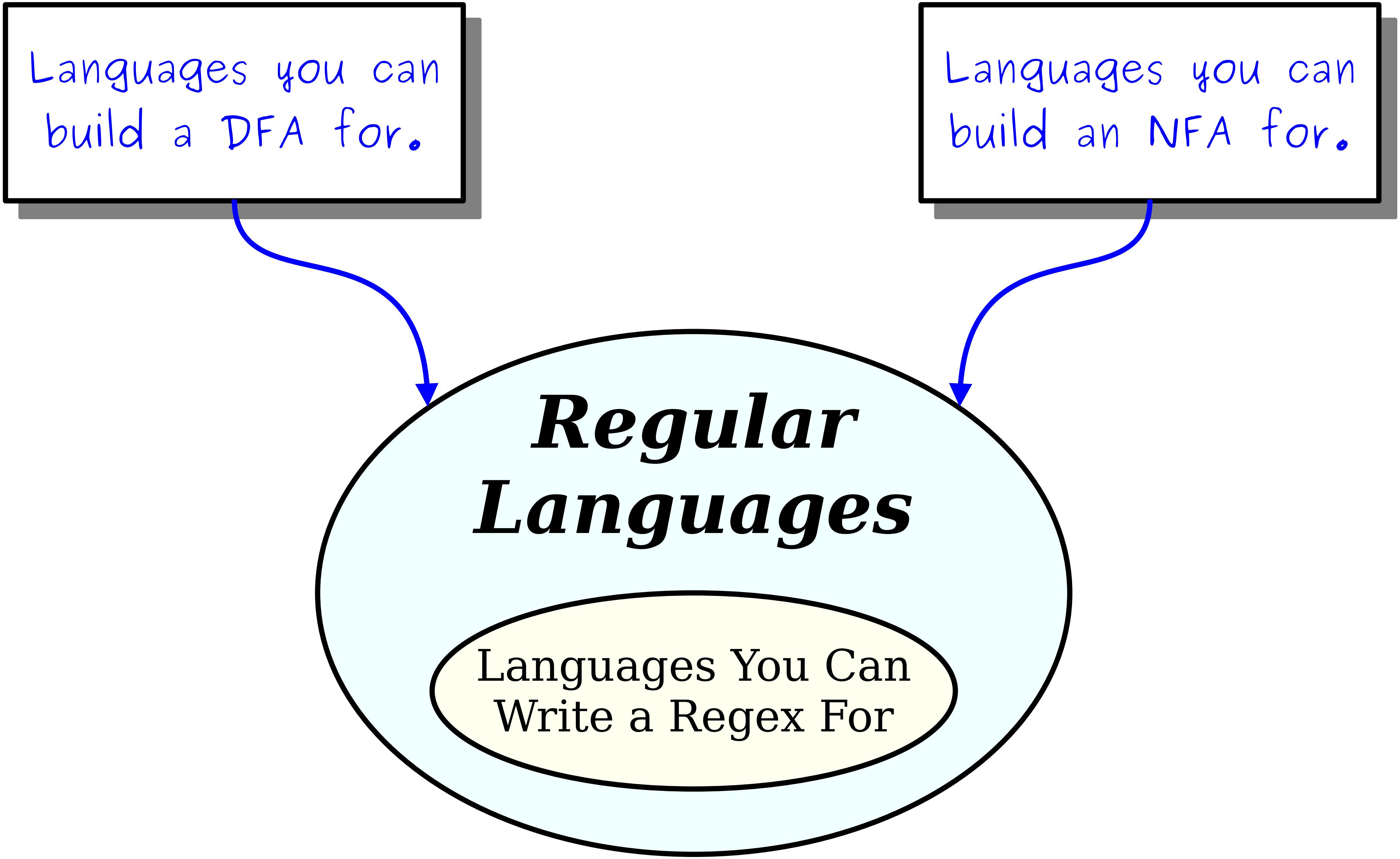
Proof idea: Use induction!

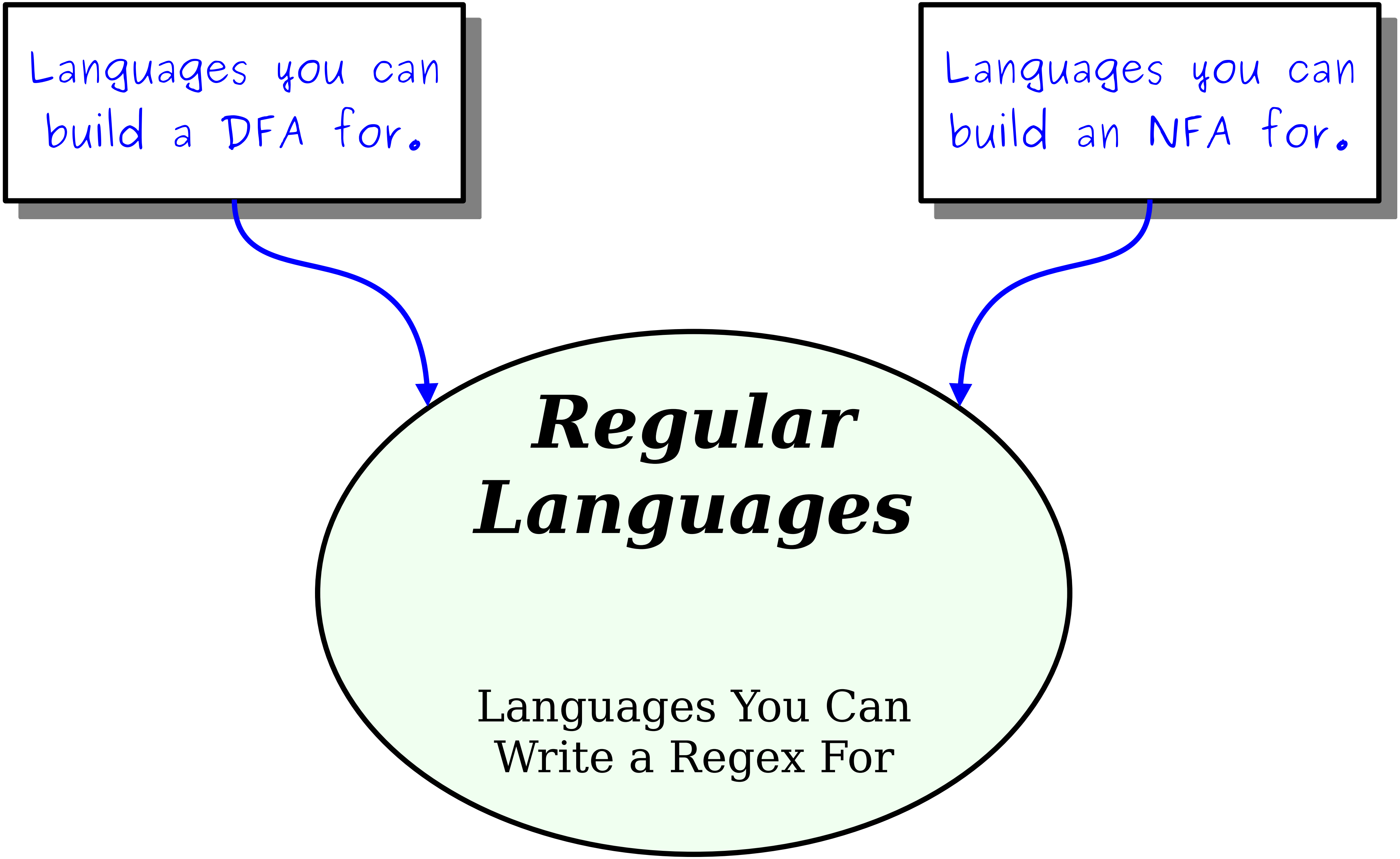
- The atomic regular expressions all represent regular languages.
- The combination steps represent closure properties.
- So anything you can make from them must be regular!

Thompson’s Algorithm

- In practice, many regex matchers use an algorithm called ***Thompson's algorithm*** to convert regular expressions into NFAs (and, from there, to DFAs).
  - Read Sipser if you’re curious!
- ***Fun fact:*** the “Thompson” here is Ken Thompson, one of the co-inventors of Unix!


Languages you can build a DFA for.
Languages you can build an NFA for.
Regular Languages


Languages you can build a DFA for.
Languages you can build an NFA for.
Regular Languages
Languages You Can Write a Regex For


Languages you can build a DFA for.
Languages you can build an NFA for.
Regular Languages
Languages You Can Write a Regex For

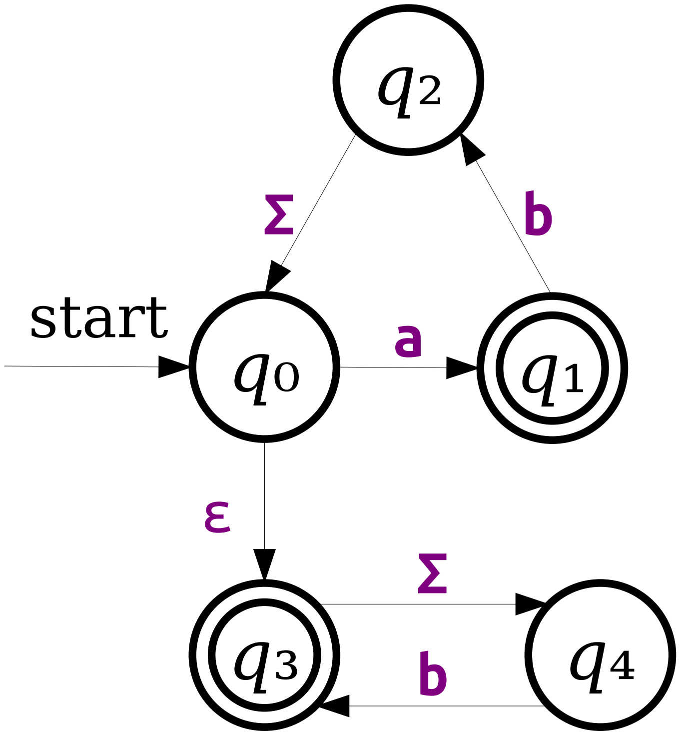
The Power of Regular Expressions

Theorem: If L is a regular language, then there is a regular expression for L.

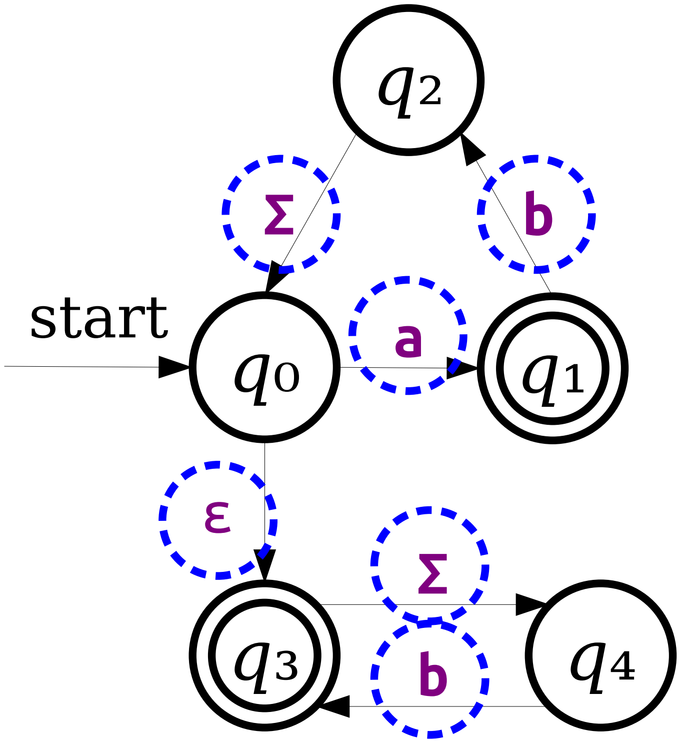
This is not obvious!

Proof idea: Show how to convert an arbitrary NFA into a regular expression.

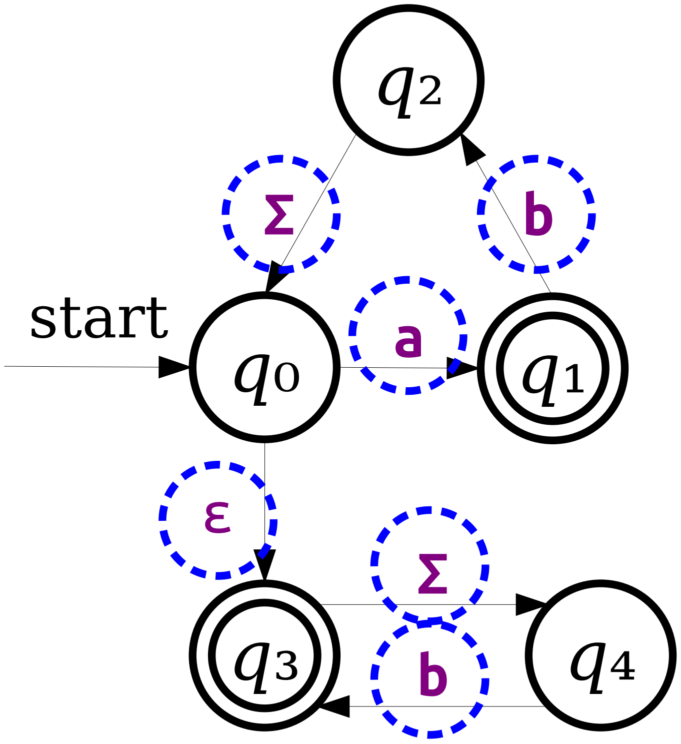
Generalizing NFAs



Generalizing NFAs

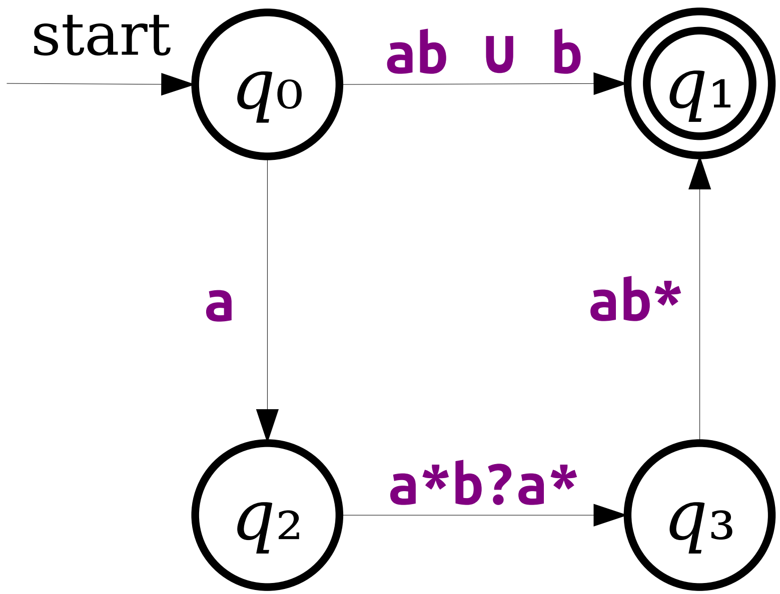


Generalizing NFAs

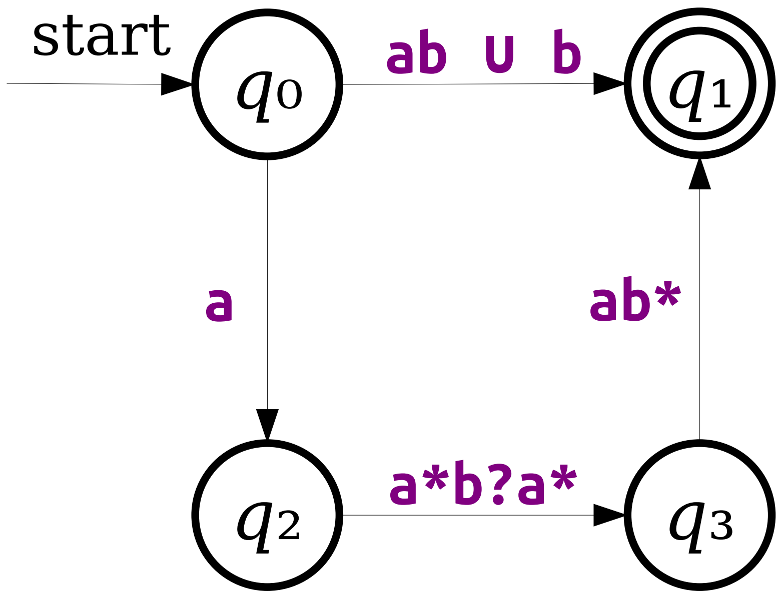


These are all regular expressions!

Generalizing NFAs

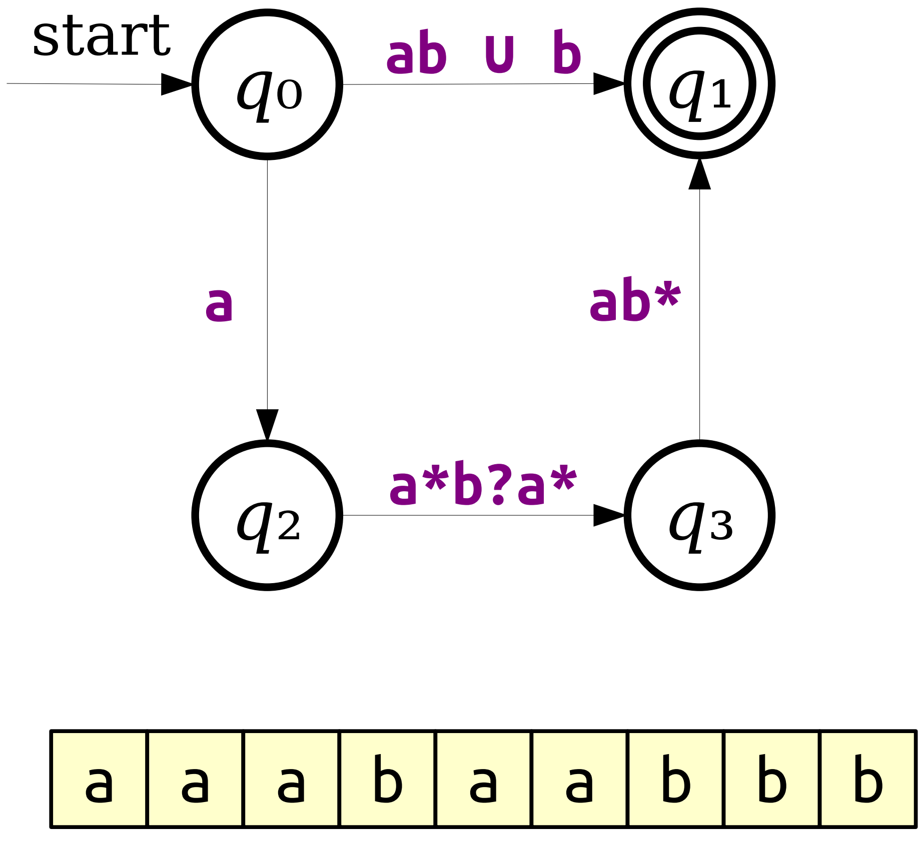


Generalizing NFAs

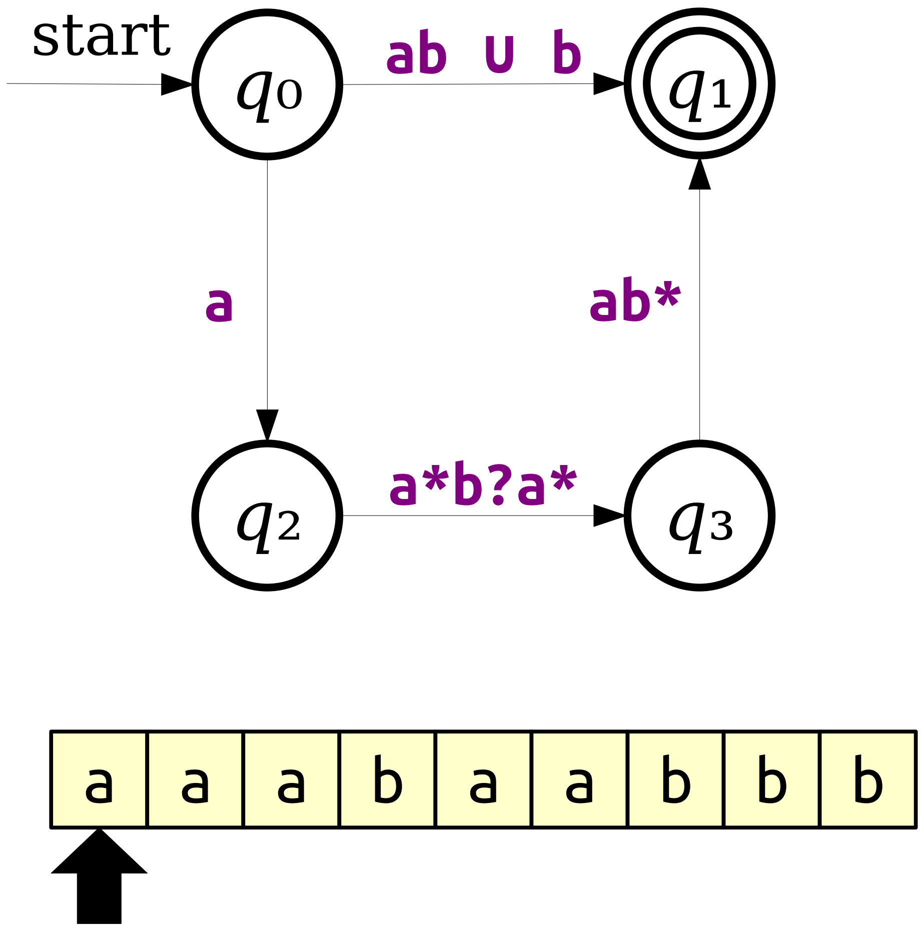


Note: Actual NFAs aren't allowed to have transitions like these. This is just a thought experiment.

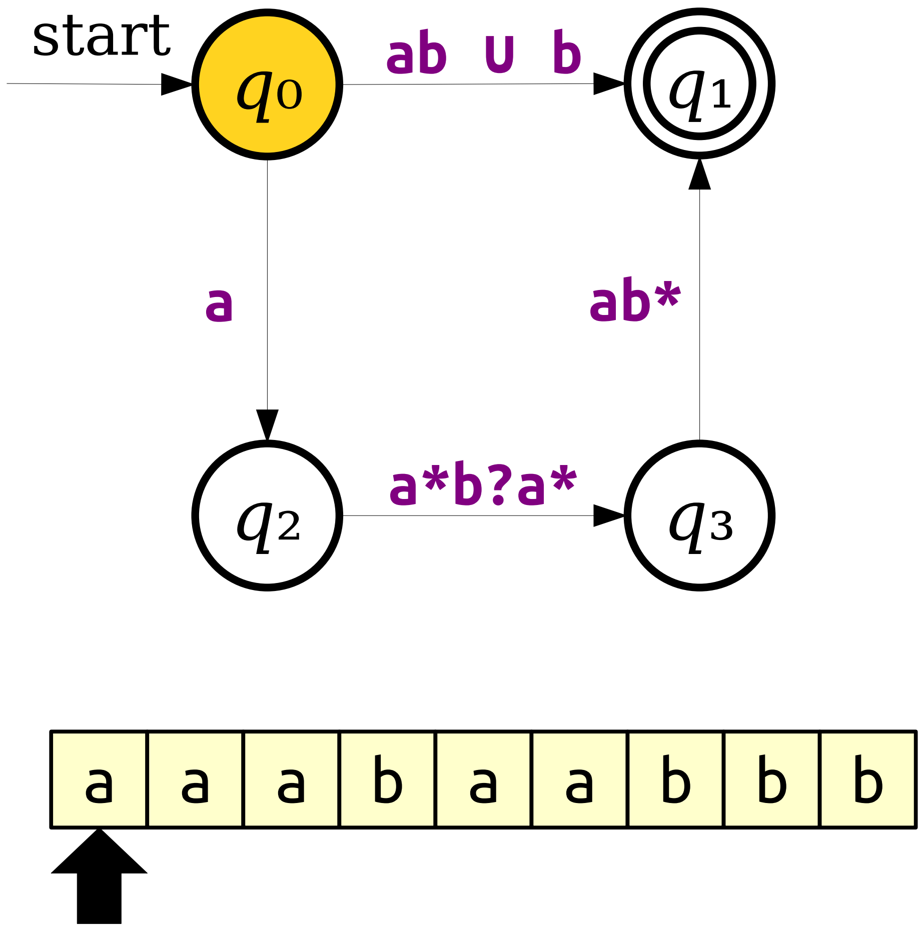
Generalizing NFAs



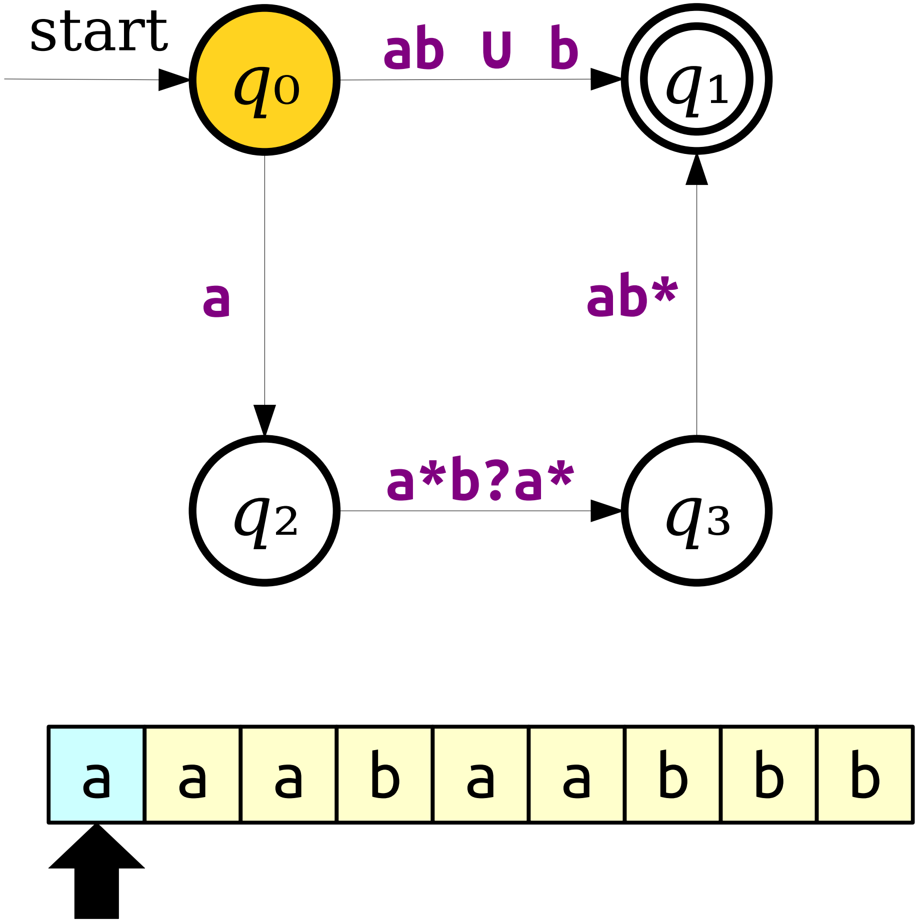
Generalizing NFAs



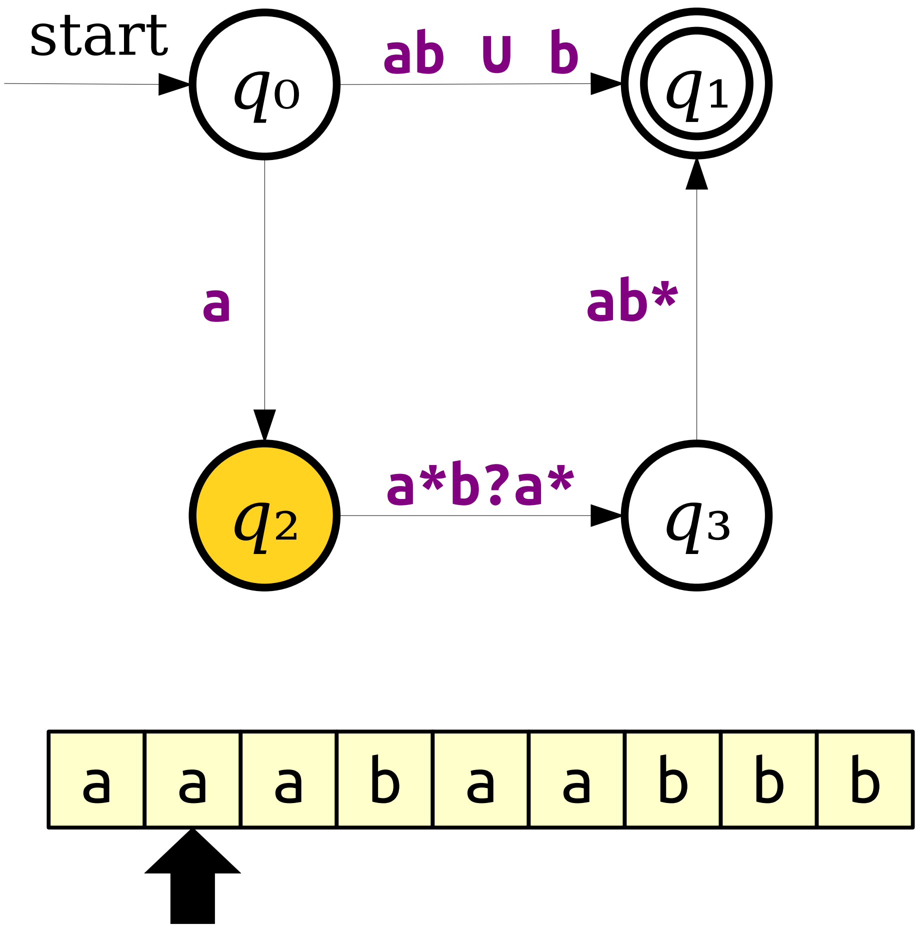
Generalizing NFAs



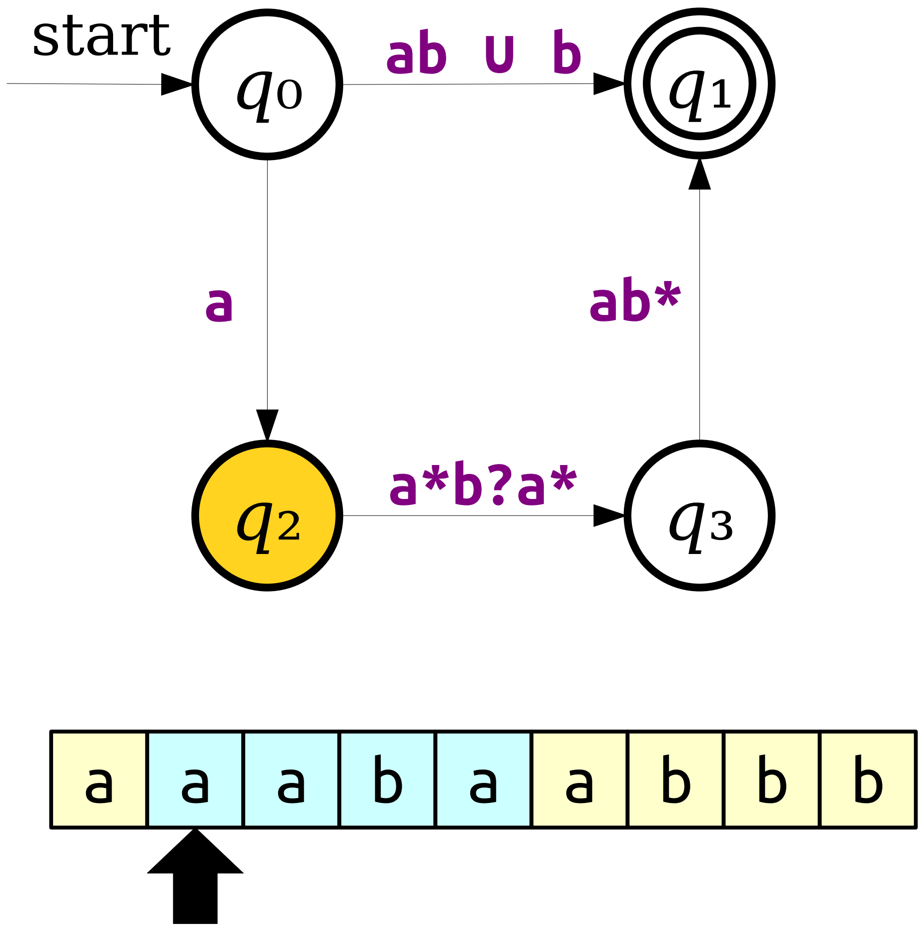
Generalizing NFAs



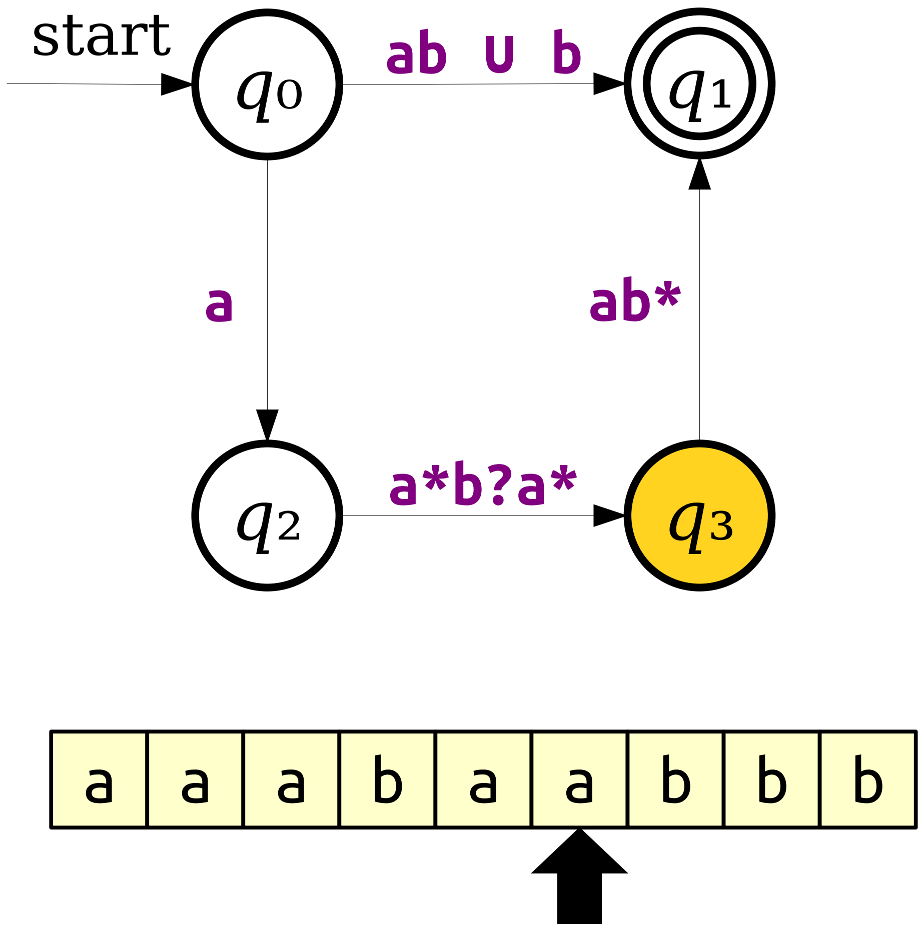
Generalizing NFAs



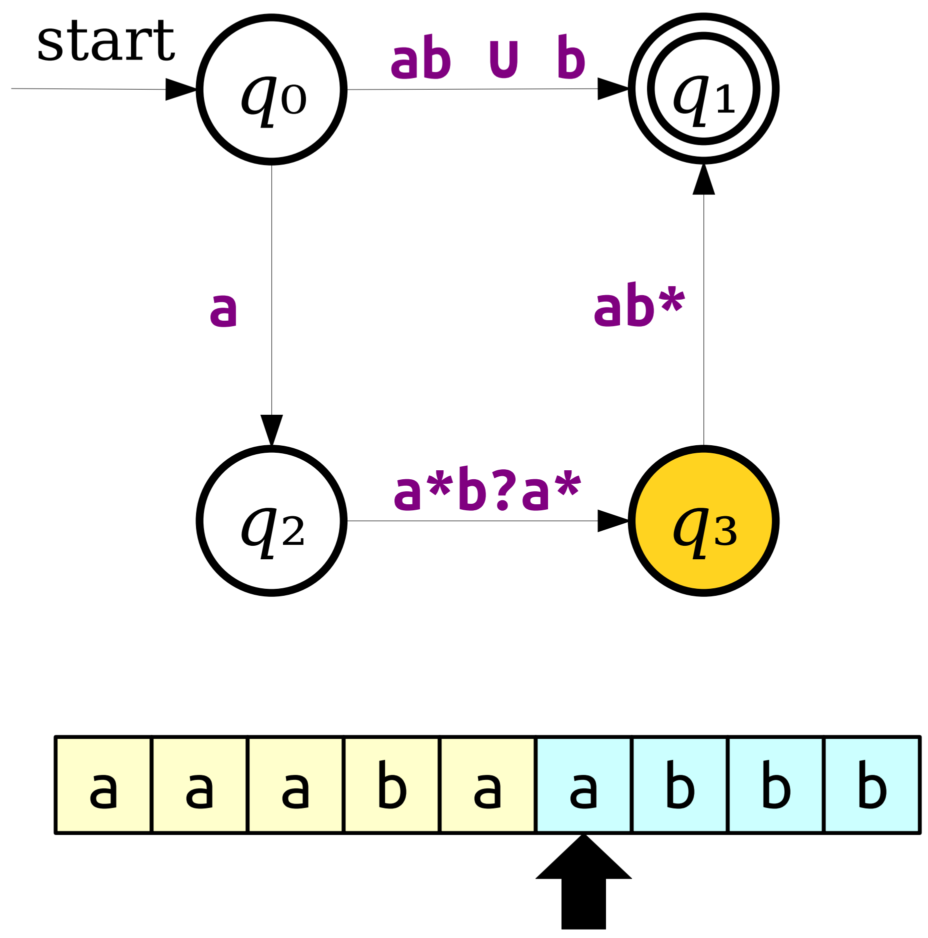
Generalizing NFAs



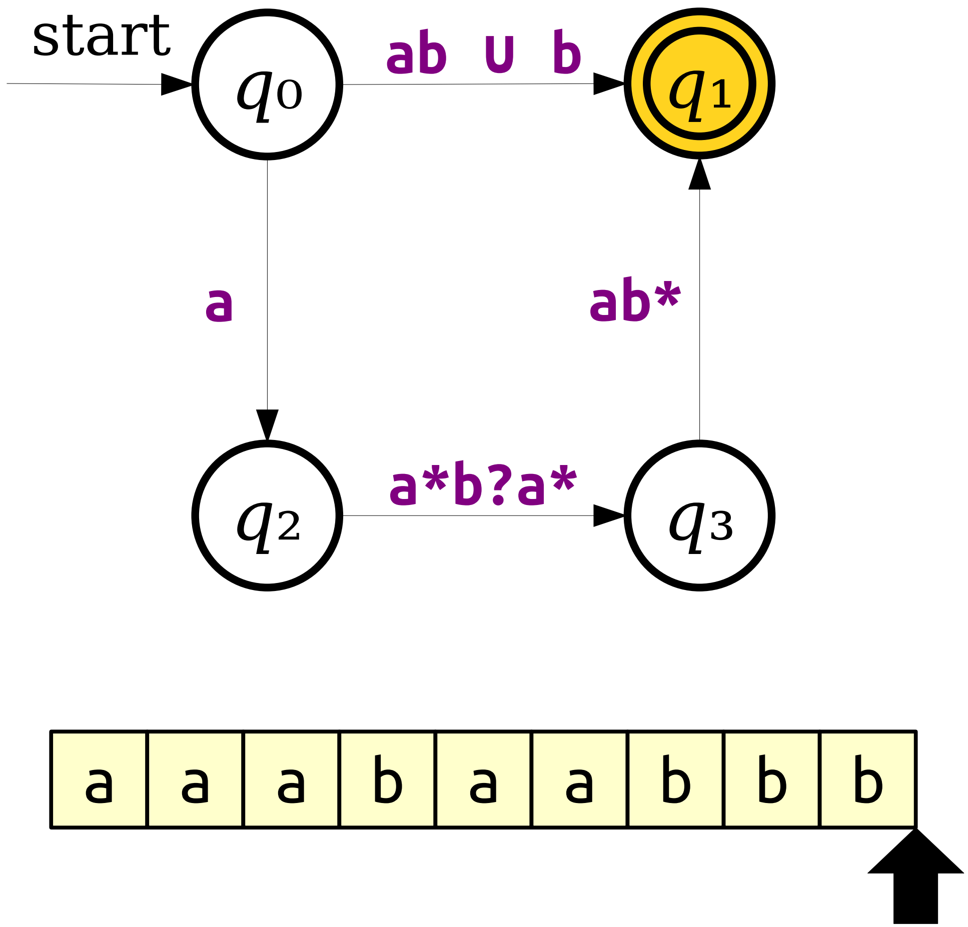
Generalizing NFAs



Generalizing NFAs



Generalizing NFAs



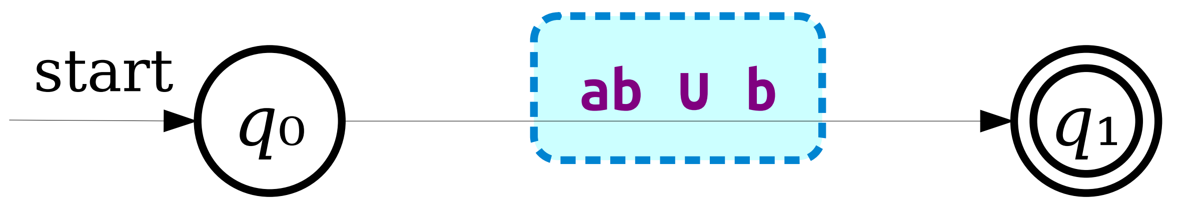
Key Idea 1: Imagine that we can label transitions in an NFA with arbitrary regular expressions.

Generalizing NFAs

Generalizing NFAs

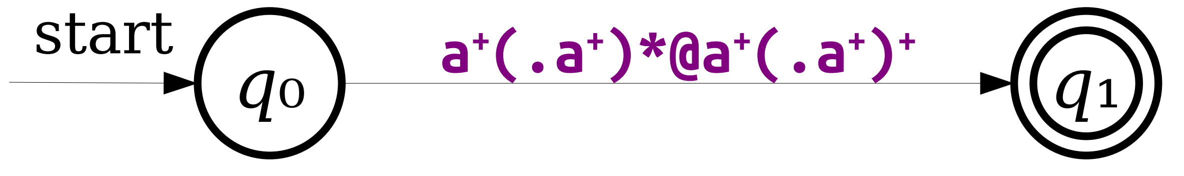
Is there a simple regular expression for the language of this generalized NFA?

Generalizing NFAs

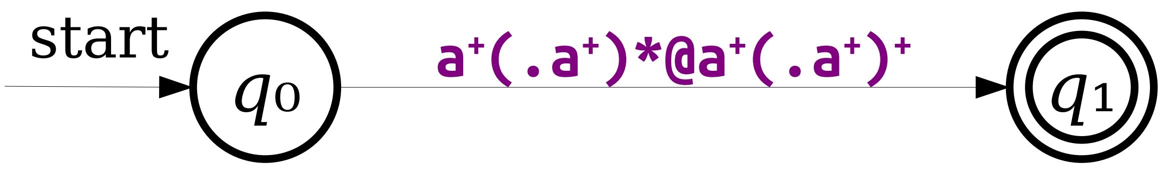


Is there a simple regular expression for the language of this generalized NFA?

Generalizing NFAs

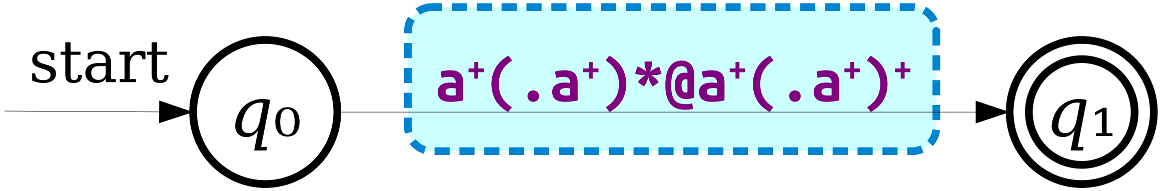


Generalizing NFAs



Is there a simple regular expression for the language of this generalized NFA?

Generalizing NFAs

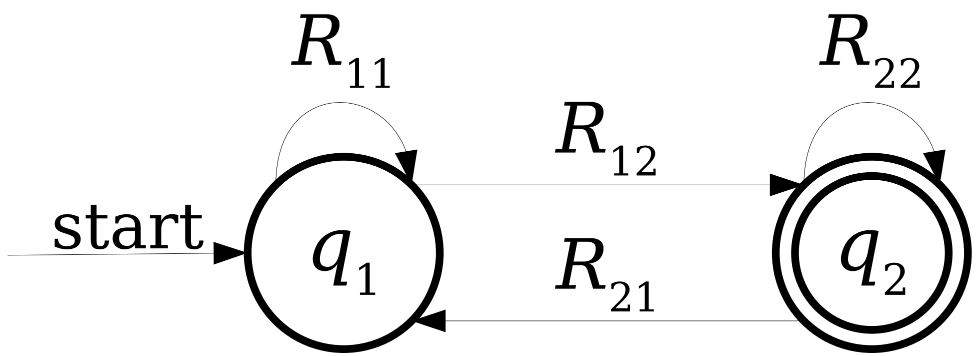


Is there a simple regular expression for the language of this generalized NFA?

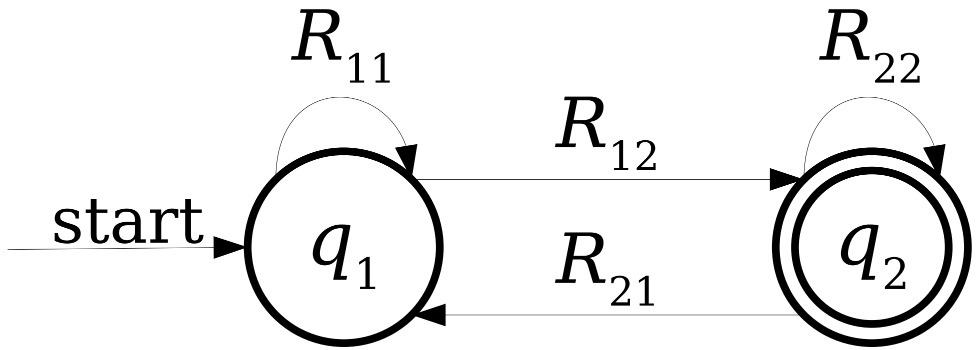
Key Idea 2: If we can convert an NFA into a generalized NFA that looks like this…

…then we can easily read off a regular expression for the original NFA.

From NFAs to Regular Expressions

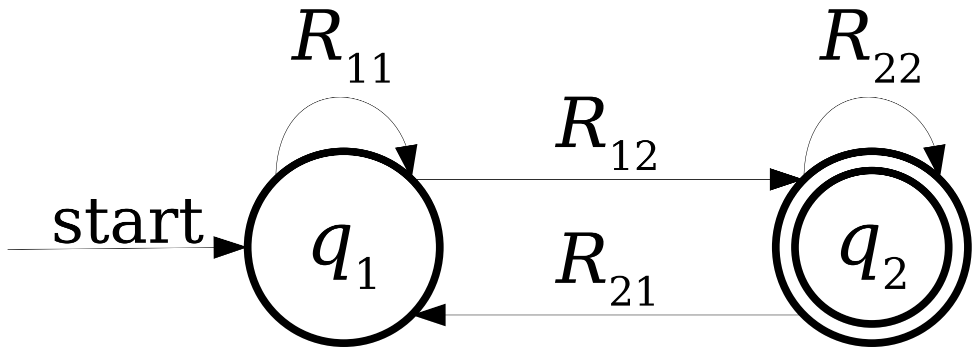


From NFAs to Regular Expressions



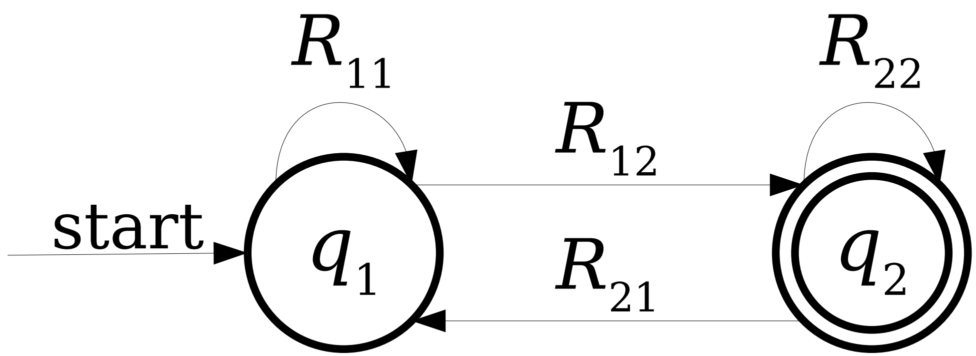
Here, R_{11}, R_{12}, R_{21}, and R_{22} are arbitrary regular expressions.

From NFAs to Regular Expressions



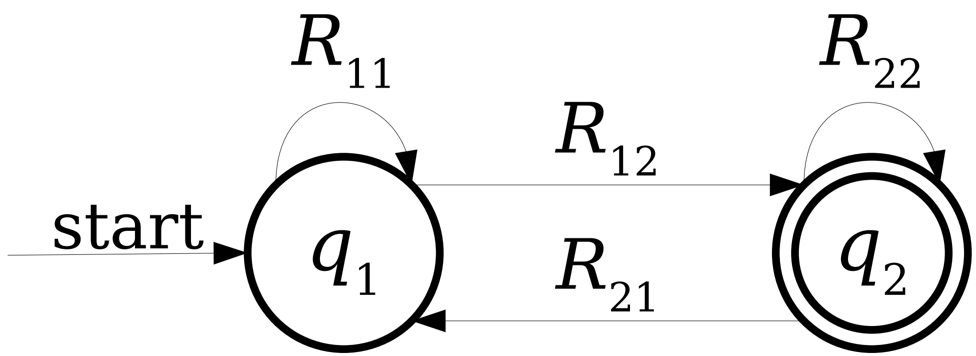
Question: Can we get a clean regular expression from this NFA?

From NFAs to Regular Expressions



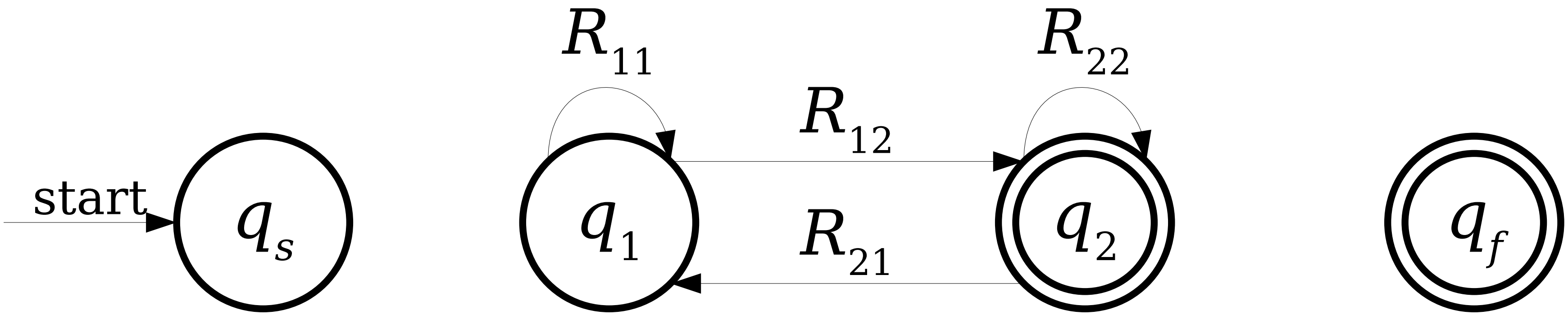
Key Idea 3: Somehow transform this NFA so that it looks like this:

From NFAs to Regular Expressions

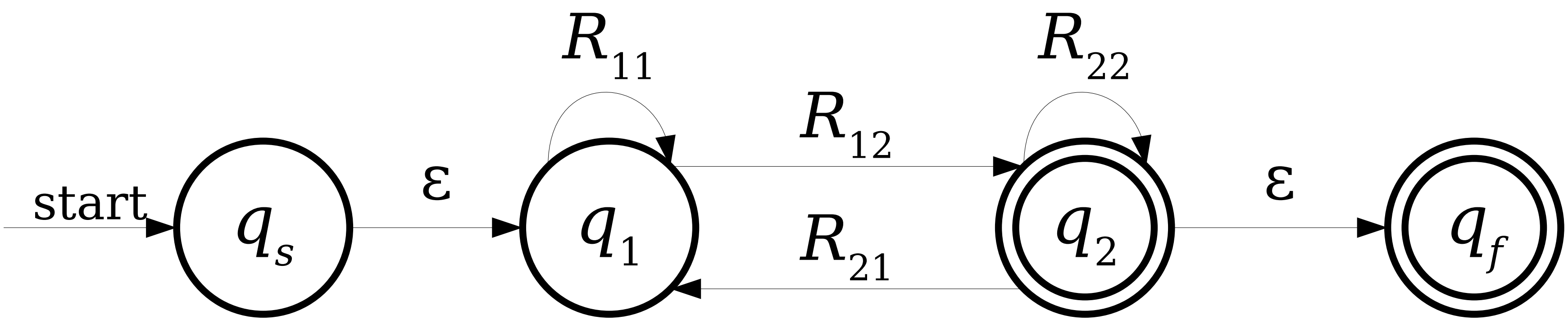


The first step is going to be a bit weird...

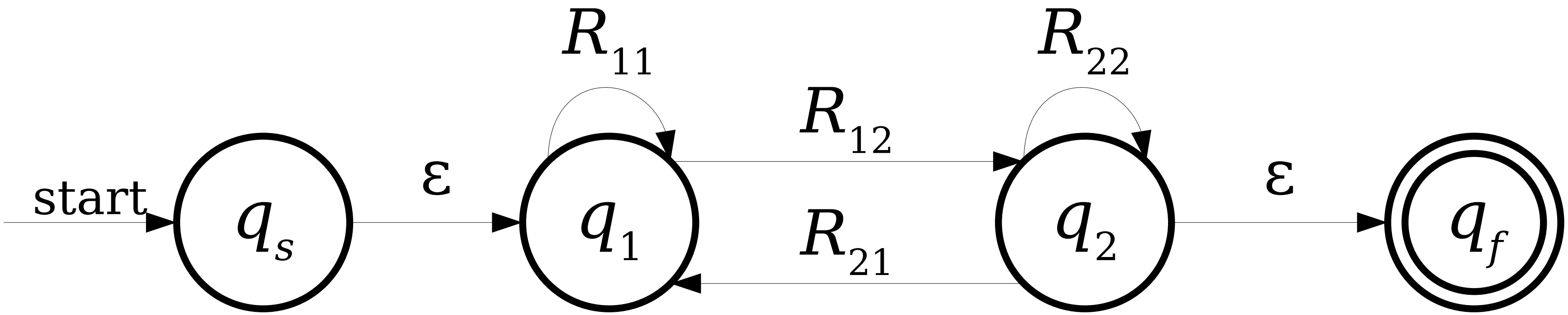
From NFAs to Regular Expressions



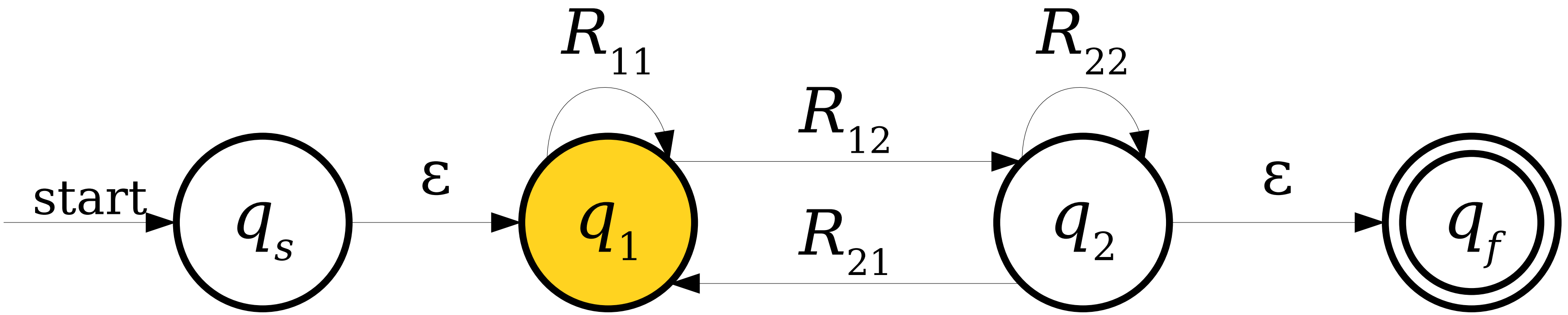
From NFAs to Regular Expressions



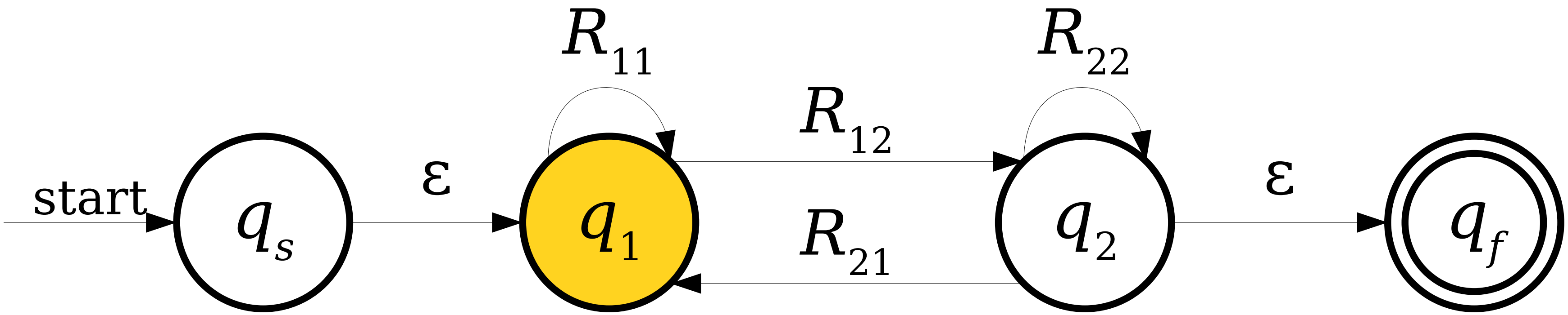
From NFAs to Regular Expressions



From NFAs to Regular Expressions

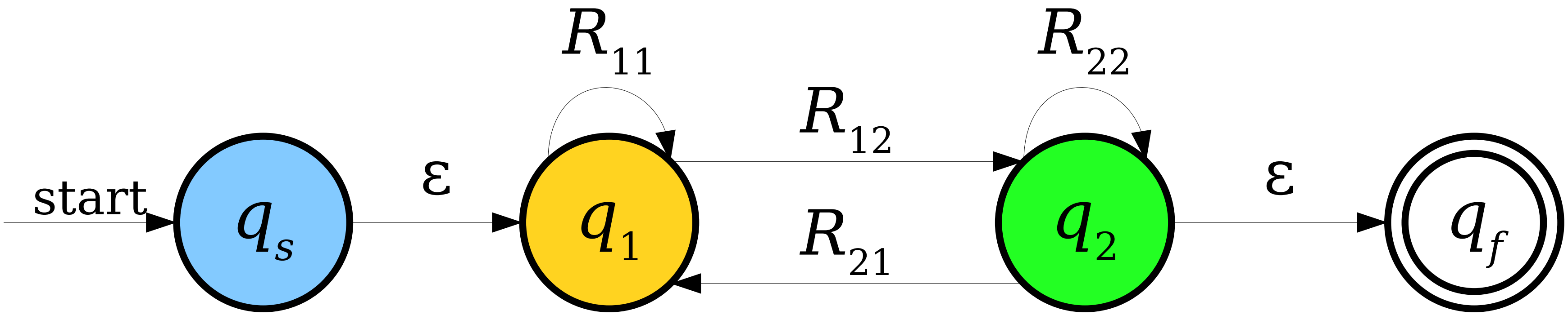


From NFAs to Regular Expressions

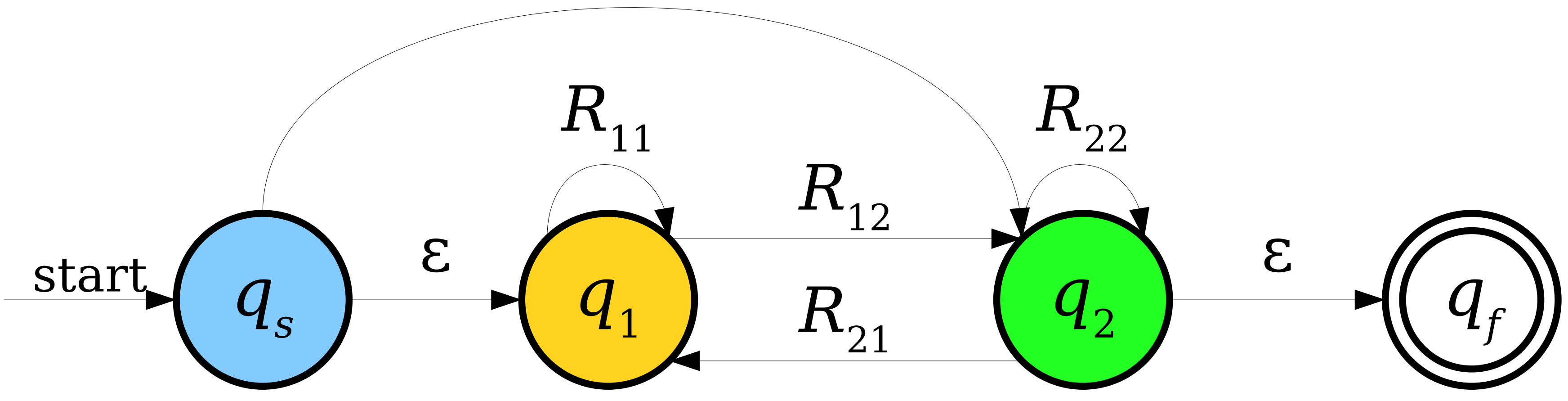


Could we eliminate this state from the NFA?

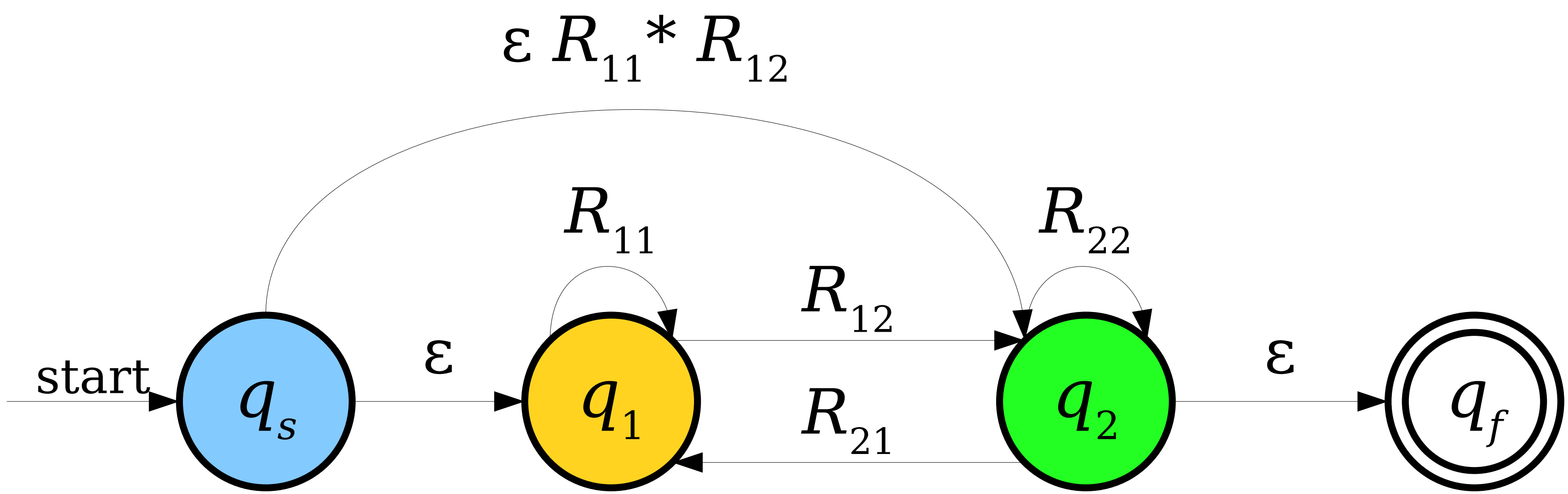
From NFAs to Regular Expressions



From NFAs to Regular Expressions

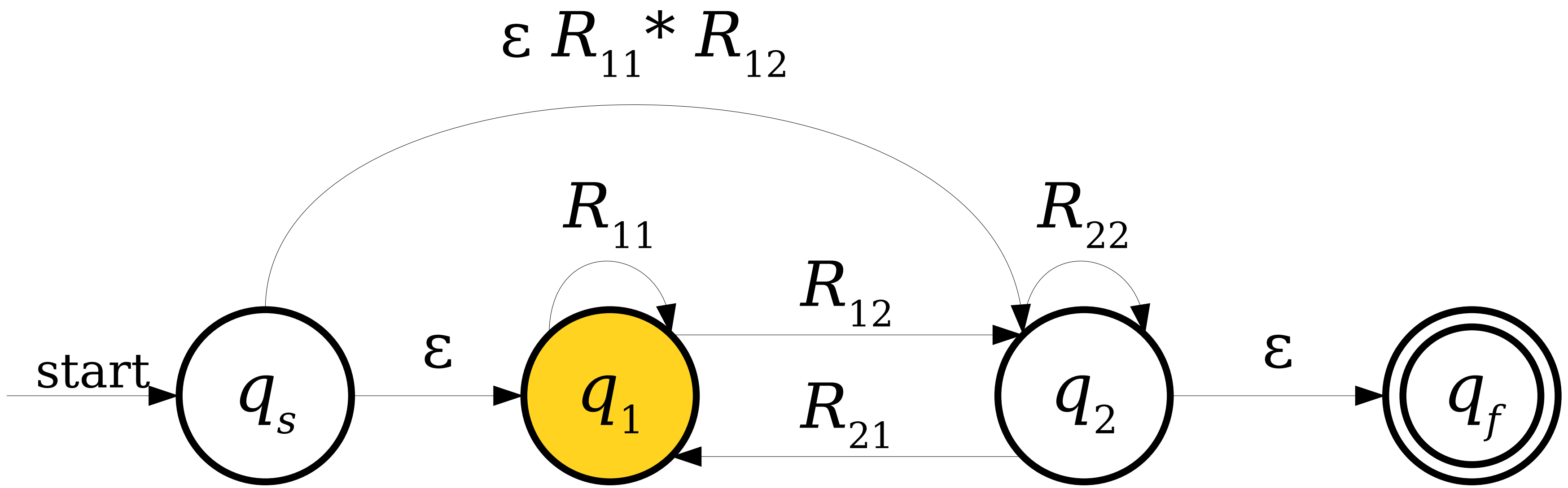


From NFAs to Regular Expressions

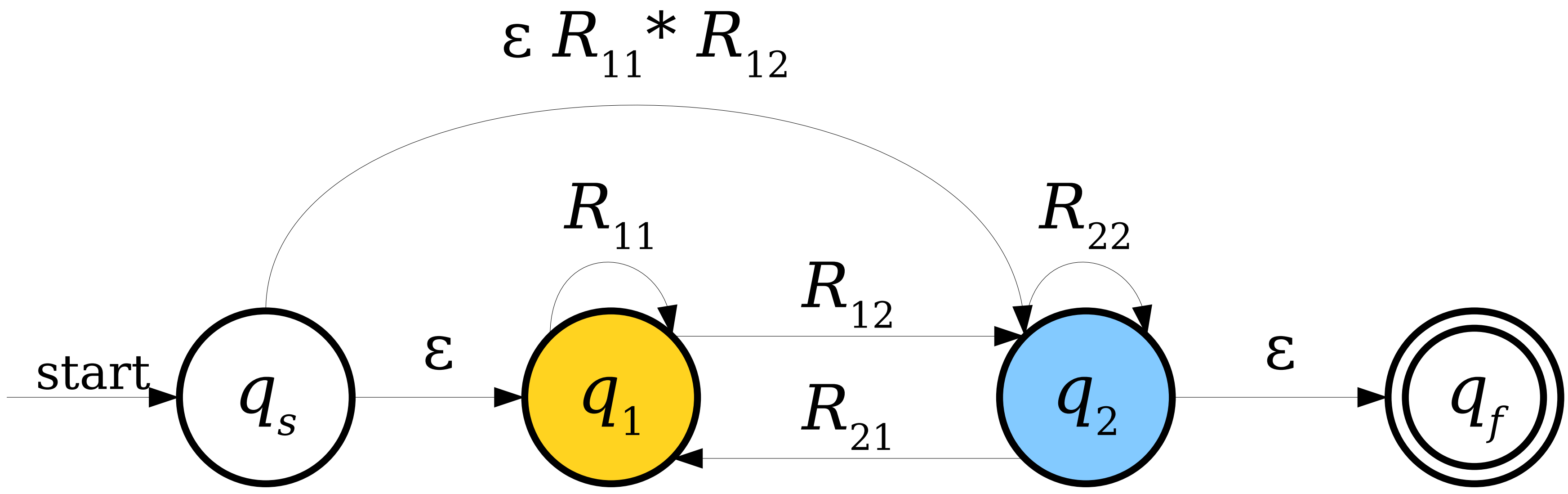


Note: We're using **concatenation** and **Kleene closure** in order to skip this state.

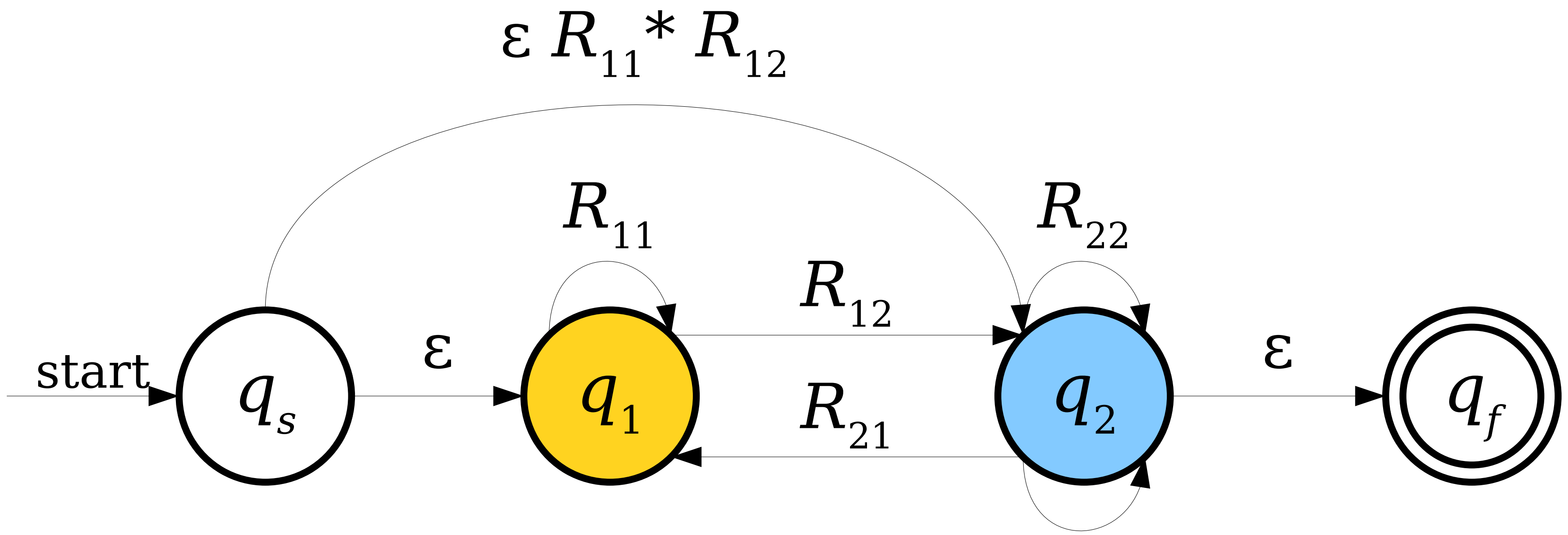
From NFAs to Regular Expressions



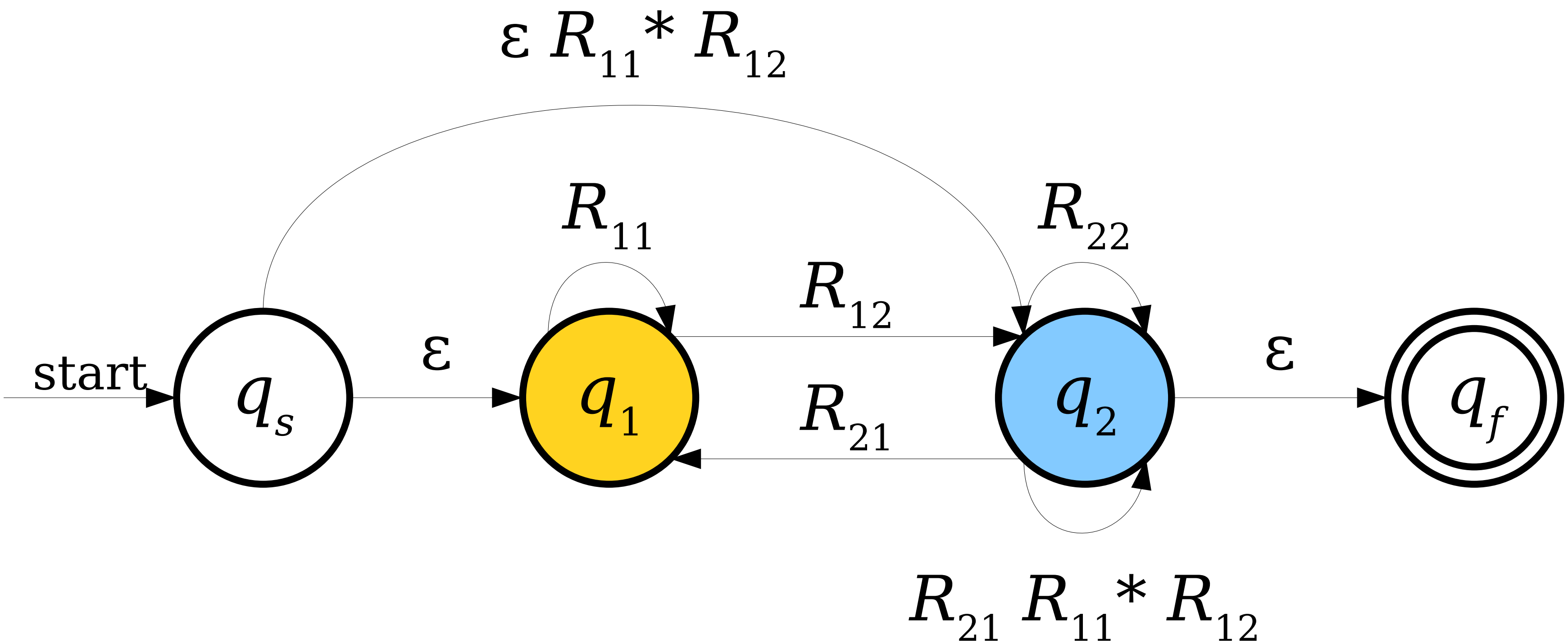
From NFAs to Regular Expressions



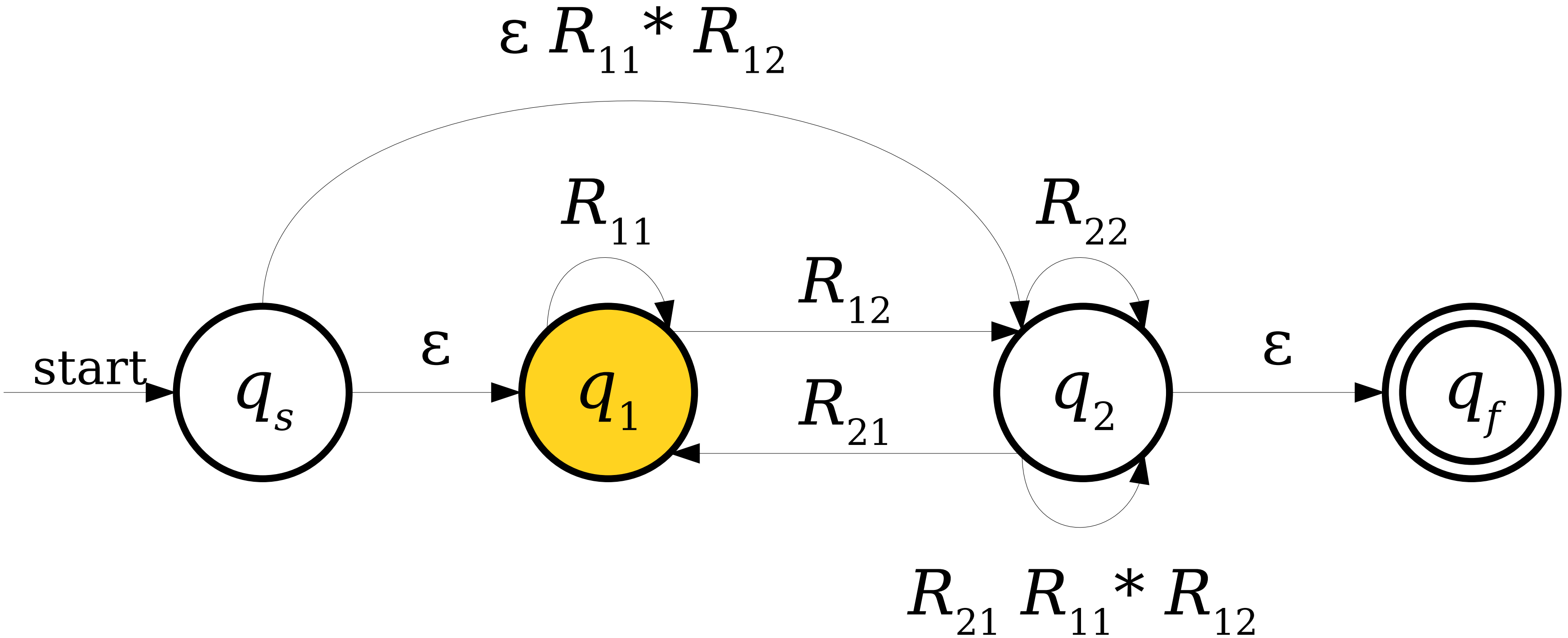
From NFAs to Regular Expressions



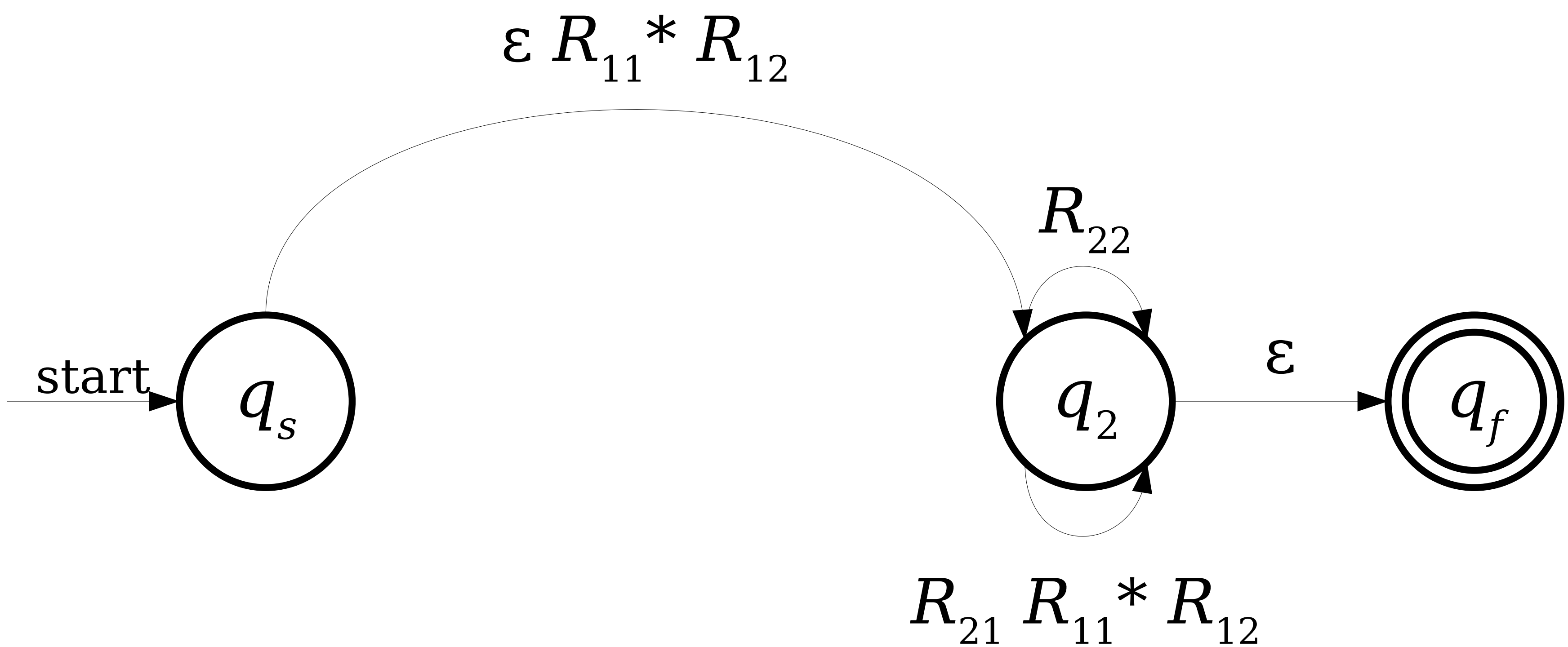
From NFAs to Regular Expressions



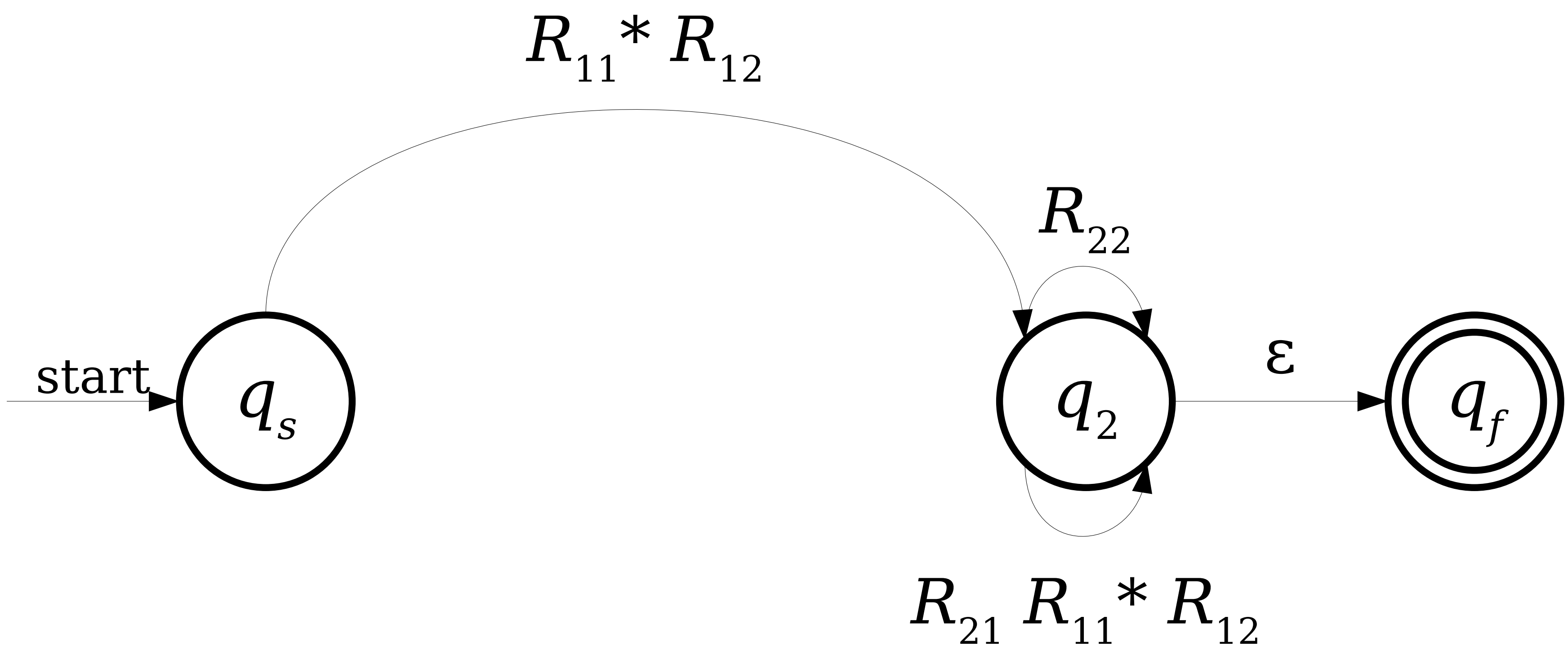
From NFAs to Regular Expressions



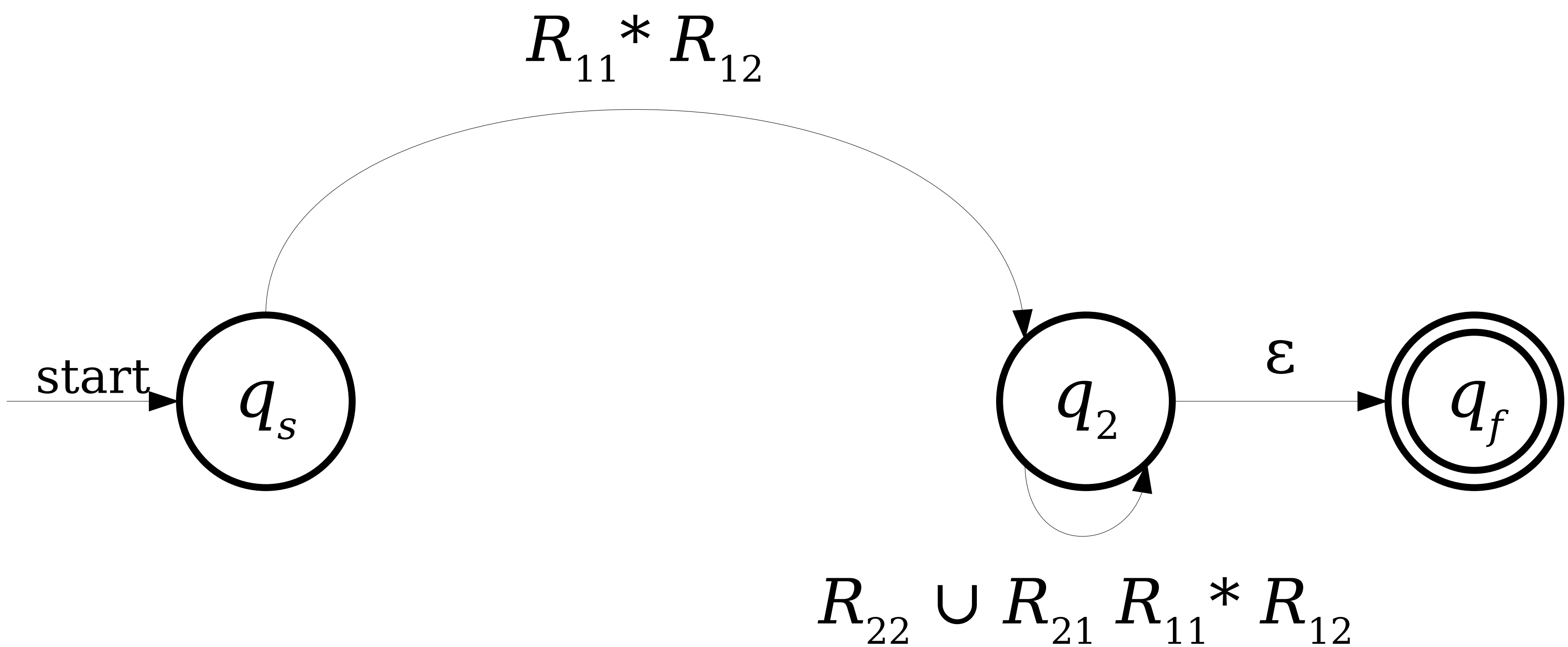
From NFAs to Regular Expressions



From NFAs to Regular Expressions

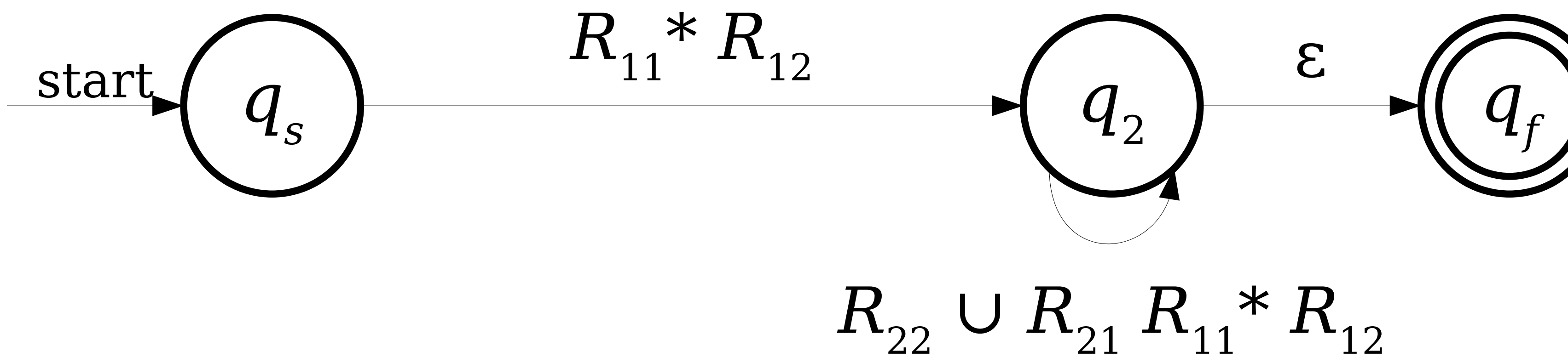


From NFAs to Regular Expressions

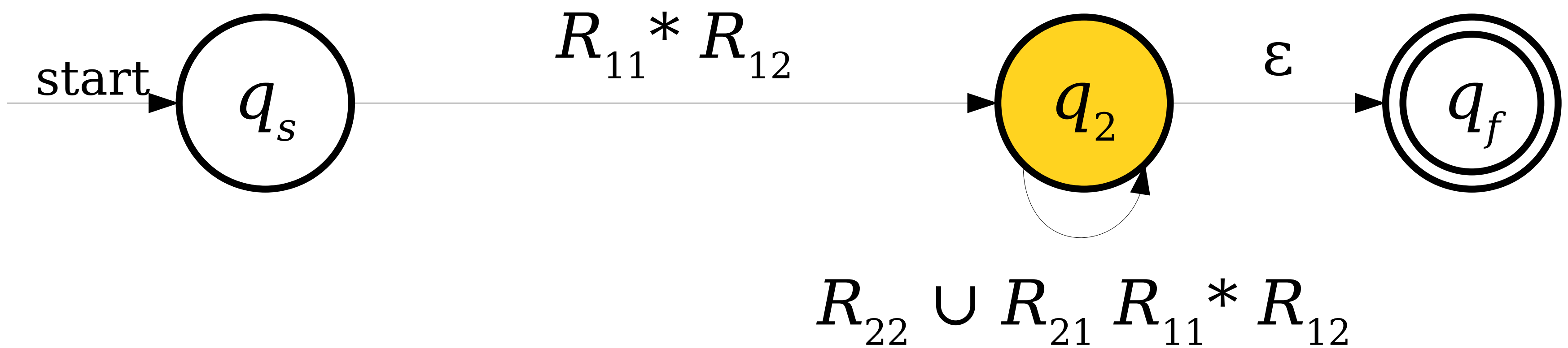


Note: We're using union to combine these transitions together.

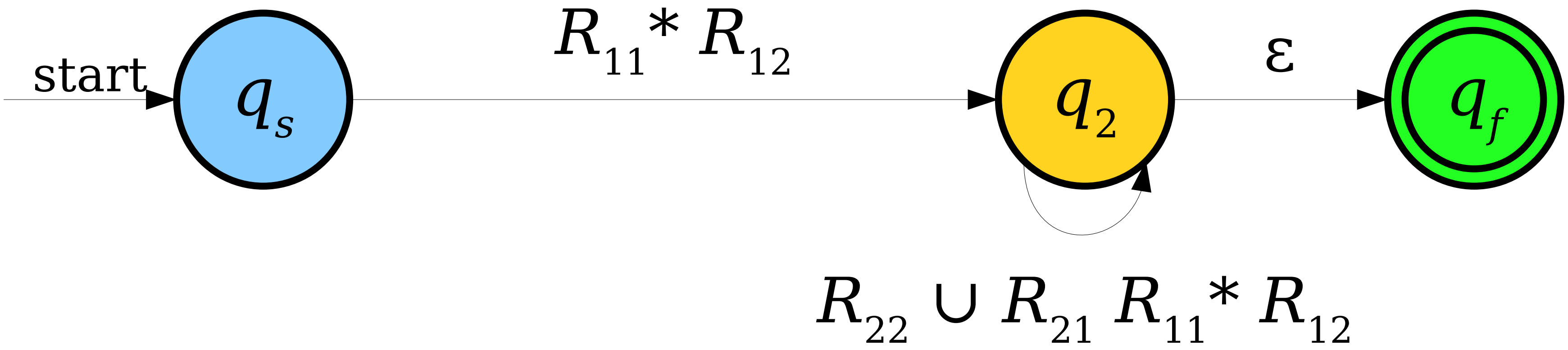
From NFAs to Regular Expressions



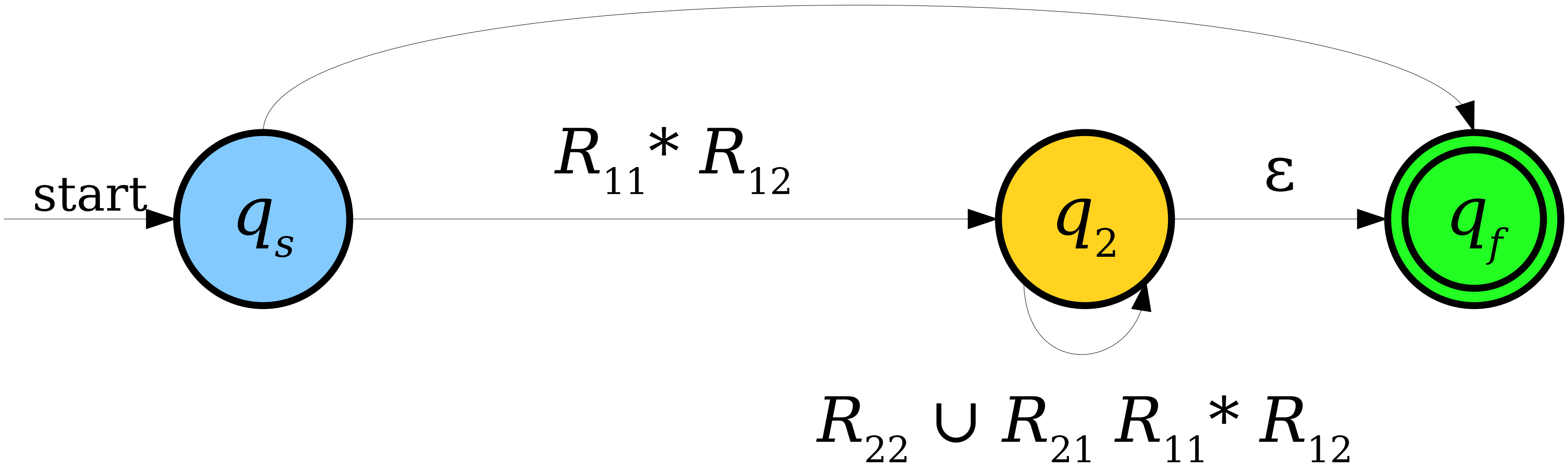
From NFAs to Regular Expressions



From NFAs to Regular Expressions

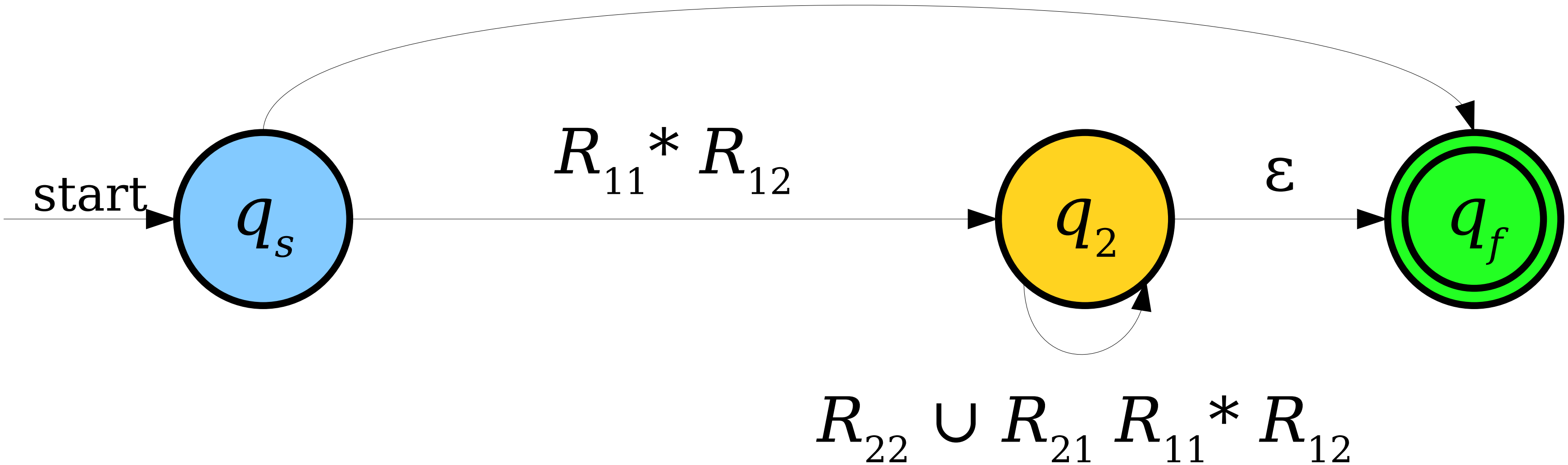


From NFAs to Regular Expressions



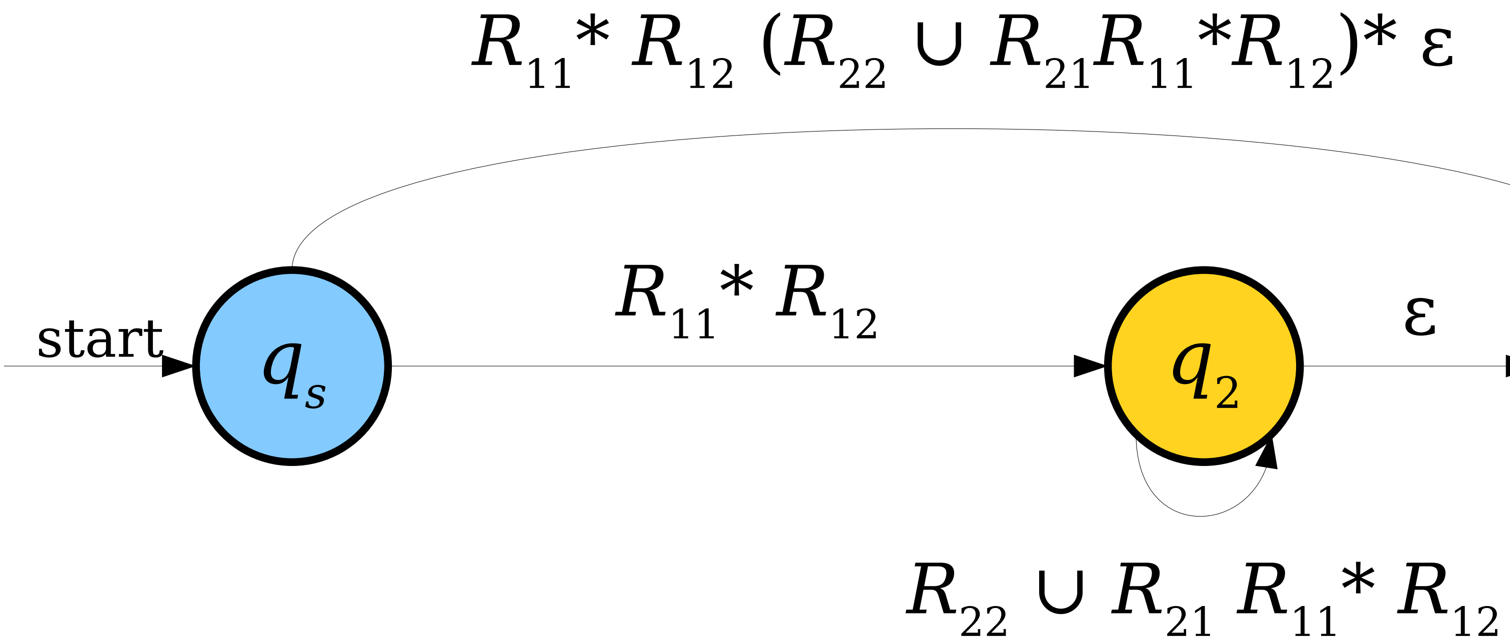
From NFAs to Regular Expressions

What should we put on this transition?

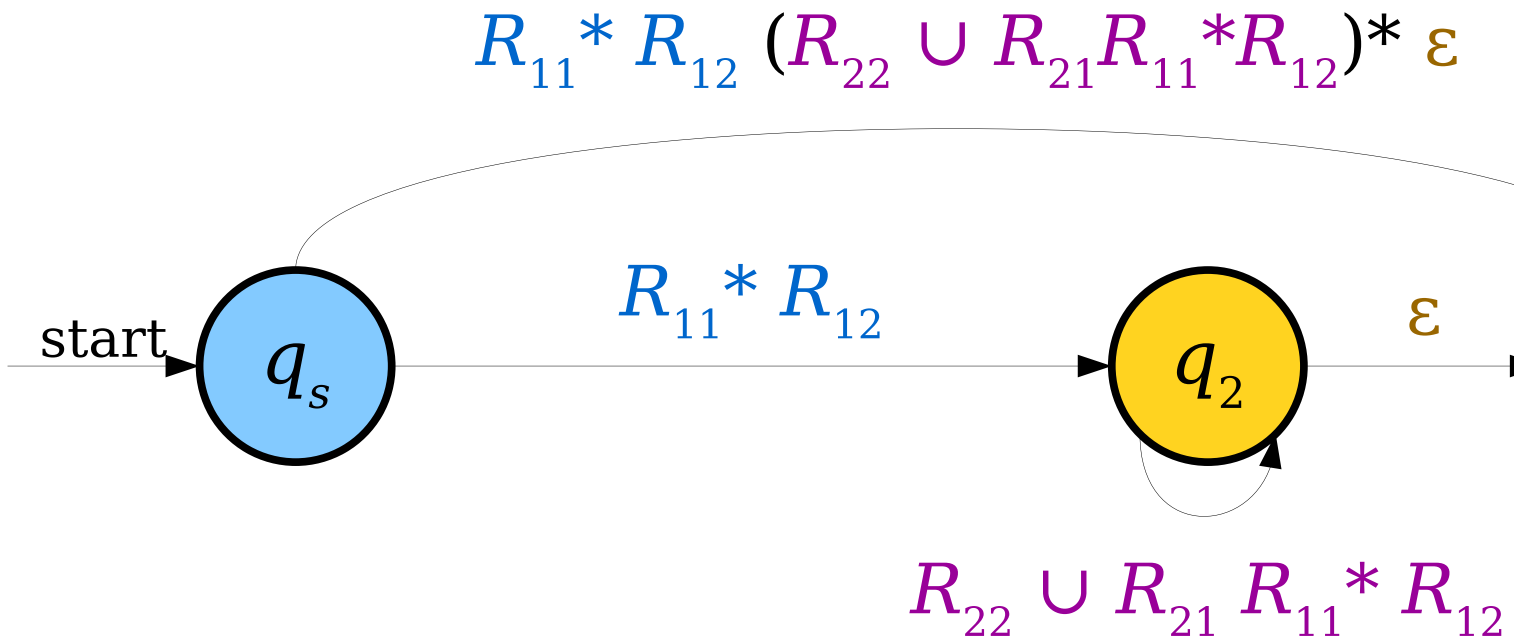


Answer at
https://pollev.com/cs103aut23

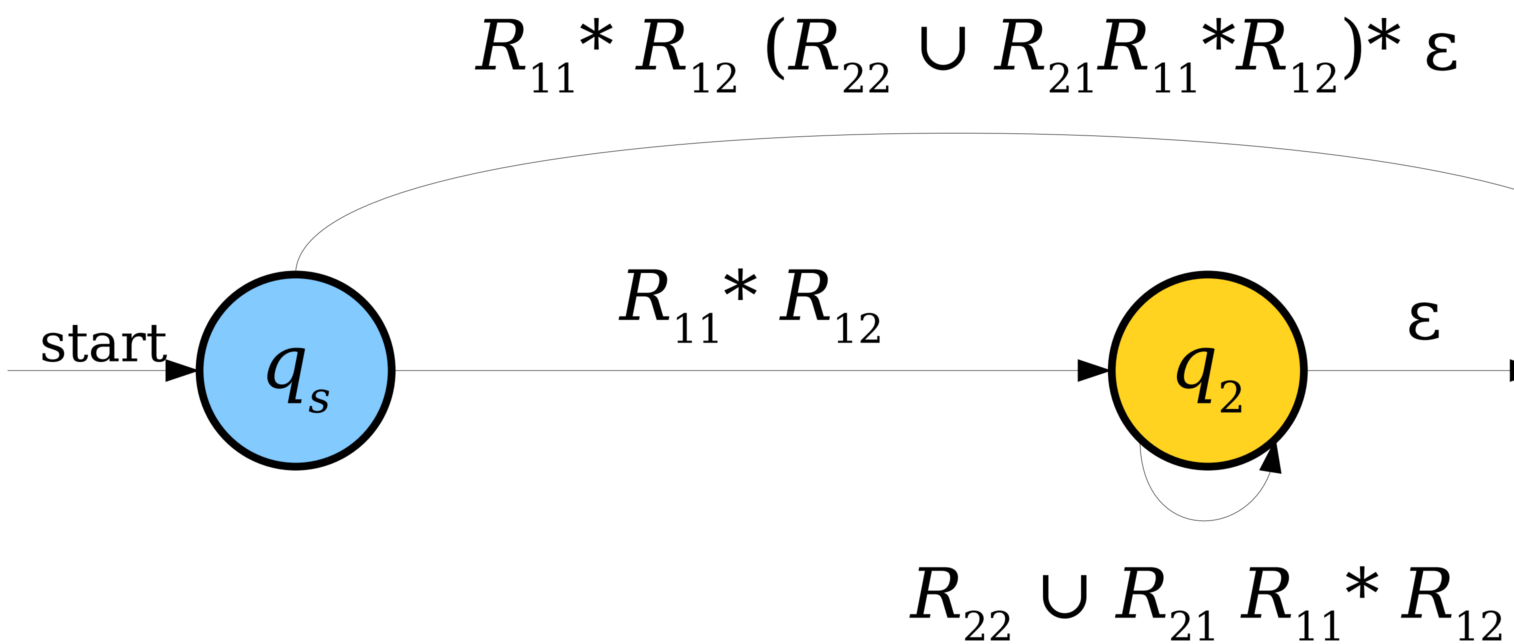
From NFAs to Regular Expressions



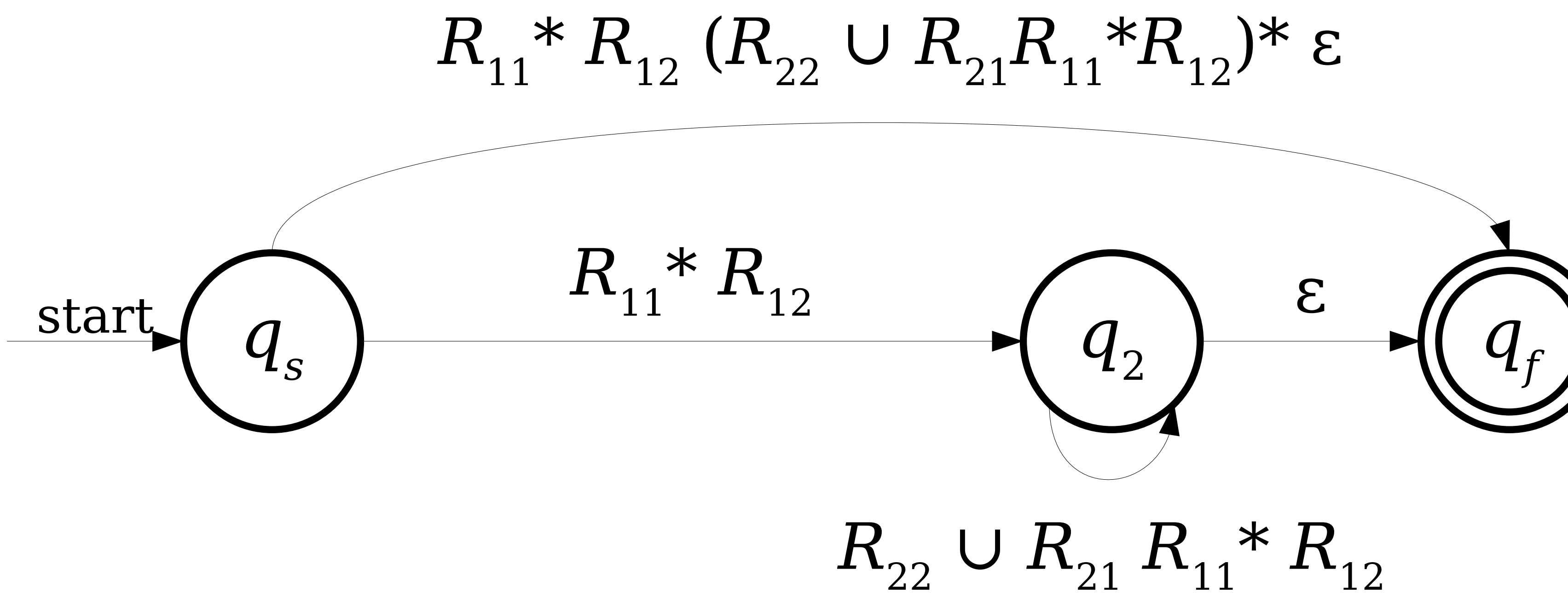
From NFAs to Regular Expressions



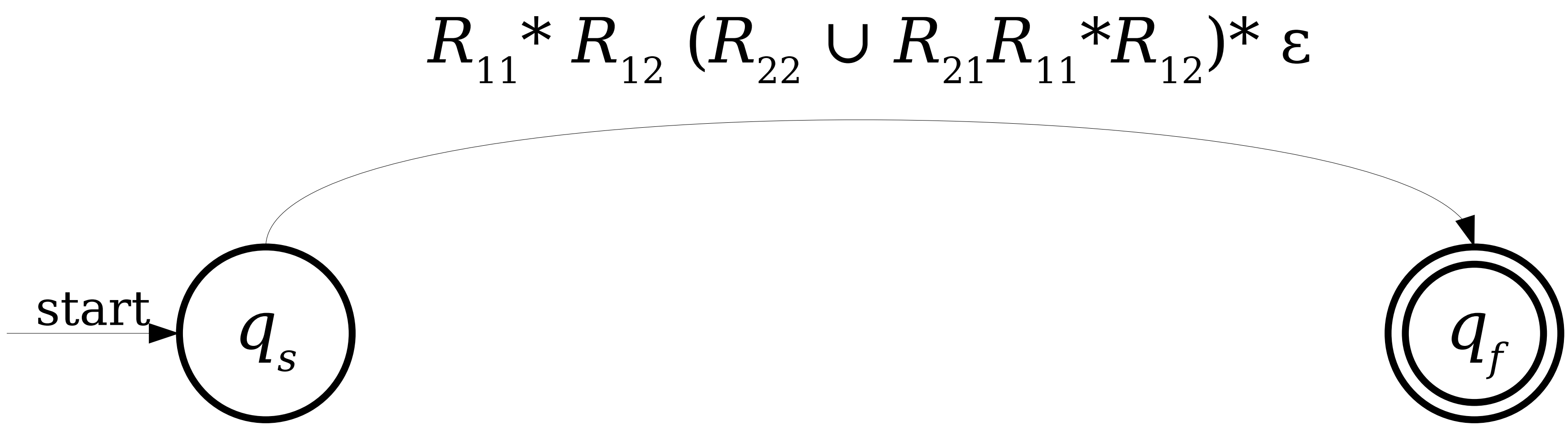
From NFAs to Regular Expressions



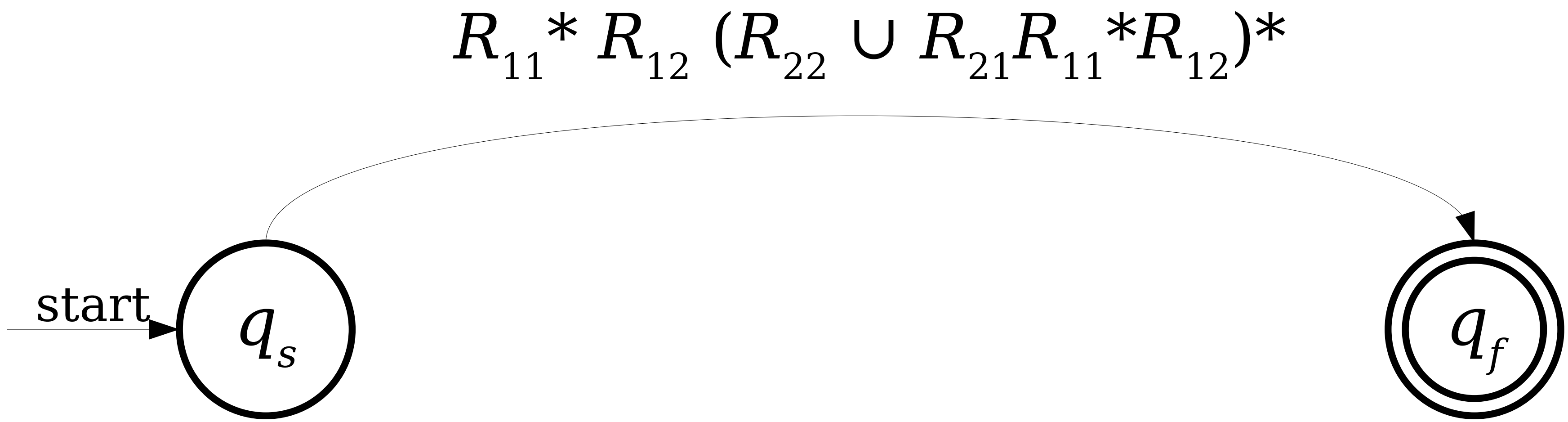
From NFAs to Regular Expressions



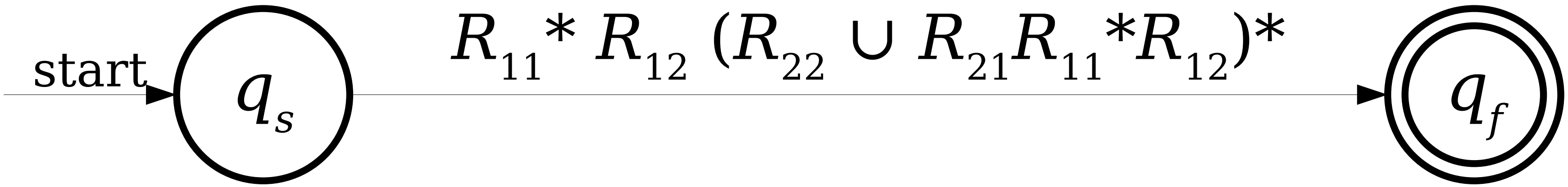
From NFAs to Regular Expressions



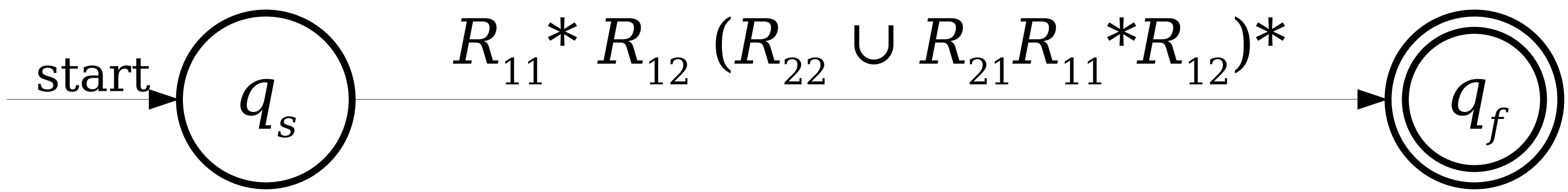
From NFAs to Regular Expressions

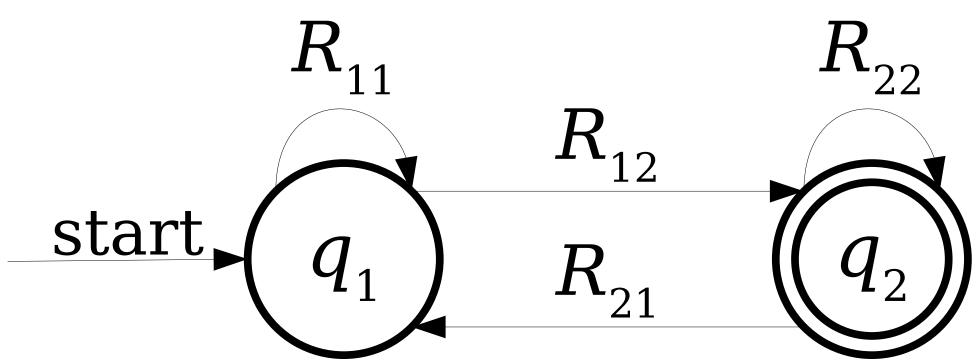


From NFAs to Regular Expressions



From NFAs to Regular Expressions





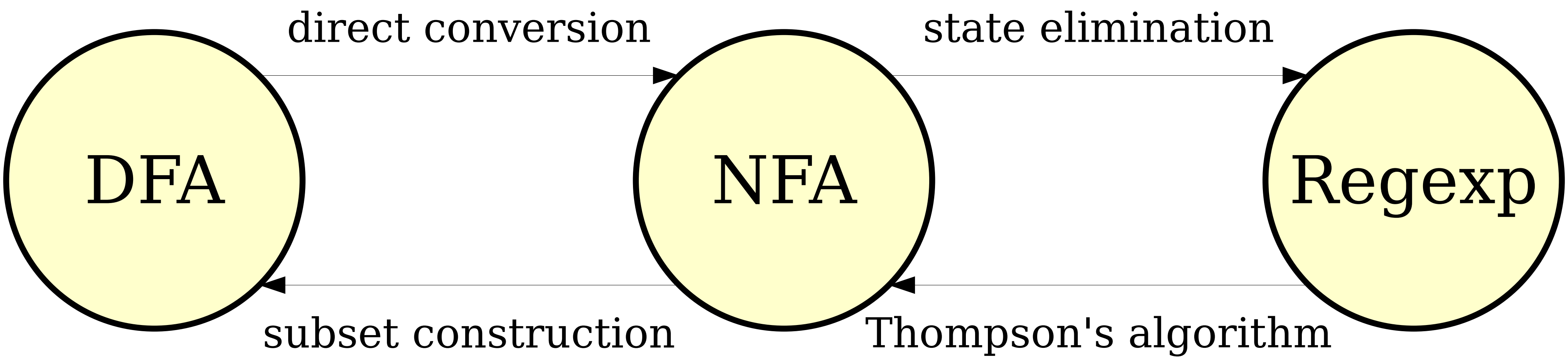
The State-Elimination Algorithm

- Start with an NFA N for the language L.
- Add a new start state q_{s} and accept state q_{f} to the NFA.
  - Add an ε-transition from q_{s} to the old start state of N.
  - Add ε-transitions from each accepting state of N to q_{f}, then mark them as not accepting.
- Repeatedly remove states other than q_{s} and q_{f} from the NFA by "shortcutting" them until only two states remain: q_{s} and q_{f}.
- The transition from q_{s} to q_{f} is then a regular expression for the NFA.

The State-Elimination Algorithm

- To eliminate a state q from the automaton, do the following for each pair of states q_0 and q_1, where there's a transition from q_0 into q and a transition from q into q_1:
  - Let R_{in} be the regex on the transition from q_0 to q.
  - Let R_{out} be the regex on the transition from q to q_1.
  - If there is a regular expression R_{stay} on a transition from q to itself, add a new transition from q_0 to q_1 labeled $((R_{in})(R_{stay})^*(R_{out}))$.
  - If there isn't, add a new transition from q_0 to q_1 labeled $((R_{in})(R_{out}))$
- If a pair of states has multiple transitions between them labeled R_1, R_2, …, R_k, replace them with a single transition labeled $R_1 \cup R_2 \cup \ldots \cup R_k$.

Our Transformations



Theorem: The following are all equivalent:

- L is a regular language.
- There is a DFA D such that $\mathscr{L}(D) = L$.
- There is an NFA N such that $\mathscr{L}(N) = L$.
- There is a regular expression R such that $\mathscr{L}(R) = L$.

Why This Matters

- The equivalence of regular expressions and finite automata has practical relevance.
  - Regular expression matchers have all the power available to them of DFAs and NFAs.
- This also is hugely theoretically significant: the regular languages can be assembled “from scratch” using a small number of operations!

Next Time

- ***Intuiting Regular Languages***
  - What makes a language regular?
- ***The Myhill-Nerode Theorem***
  - The limits of regular languages.